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ESSENTIALS

OF

GEOMETRY

BY

ALFRED H. WELSH, A.M.

LATE PROFESSOR OF MATHEMATICS IN BUCHTEL COLLEGE, AUTHOR OF
DEVELOPMENT OF ENGLISH LITERATURE
AND LANGUAGE

If a man's wit be wandering, let him study the mathematiks; for in demonstrations if his thought be called ever so little away, he must begin again.—*Lord Bacon*

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TO

MY FRIEND AND BENEFACTOR,

GOVERNOR CHARLES FOSTER.

NIHIL EST TAM POPULARE QUAM BONITAS.—*Cicero.*

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PREFACE.

THE controlling purpose of the present work is not science, but education. The following principles, therefore, will be found fundamental to its design and execution:

1. That the peculiar advantages of mathematics in forming habits of close reasoning, the chief benefits to be derived from them in a harmonious development of the powers, are comprised in geometry.

2. That, while mathematical instruction *per se* is freely conceded to leave a chasm in mental culture, none can hope for the highest efficiency of his faculties without the training which this study is fitted to give, and which no other can supply.

3. That this branch possesses a twofold utility—***absolute***, so far as its cultivation is immediately conducive to mental improvement; *relative*, so far as its cultivation is necessary for the prosecution of other branches of knowledge.

4. That its most considerable absolute good is the correction of a certain vice and the formation of its corresponding virtue; the vice being the habit of mental distraction; the virtue, the habit of continuous attention.

5. That the utmost effectiveness of the study as a means of discipline requires a specific adaptation, based upon the initial idea that every beginner is intellectually an infant. Demonstrative reasoning is a mode of energizing to which he has thus far been a total stranger, and he finds himself in a new sphere of being and action. In thought, as in locomotion, he must creep before he walks, and 'be fed as he is able to bear it.' The true ideal is to render the subject intelligible, yet to avoid an enervat-

ing manner of teaching it. Doubtless it was because his book was not a class-book that Euclid gave his famous answer to the king who asked him to make geometry easier.

Accordingly, I have endeavored at the outset, by a process of induction, to fix clearly in the student's mind the distinction between the invisible entities of geometry and the visible representations of them, ignorance of which is a fruitful source of discouragement and failure. Clear perceptions must precede an intelligent exercise of the faculty of comparison.

Definitions are introduced only as they are demanded for immediate use. The learner is not held for several days to the mechanical task of memorizing a series of abstractions, and so made to acquire at the start a distaste for the whole subject, but, with every needed preparation, is brought quickly to the applications. Ill-digested and (for the time) useless information may tend to intellectual fat, but never to intellectual muscle.

Throughout, especially in the earlier chapters, questions are asked with a view to giving the student a clear idea of the essential steps in the demonstration, assisting his memory, and cultivating precision, as well as independence, of thought and expression. At suitable places, also, remarks of a suggestive character are inserted, to aid him in an economic and successful employment of his energies.

Numerical examples, selected for the application of principles to the actual business of life, as well as theorems of which original demonstration is required, are appended, with liberal suggestions, to many sections and at the end of every chapter; of sufficient difficulty to stimulate to success, yet not so difficult as to depress by failure. Exercises must be neither so multiplied in number, nor so difficult in kind, as to defeat their end. Let them be progressively adapted to the abilities of the average mind, and they will best accomplish their purpose with all,— keener interest, skill in the use of theory, and self-reliant activity.

Occasionally, in the first stages, the same theorem is required to be demonstrated from a figure differently lettered, unlettered, or differently placed. It is certainly beneficial to the student to see the truths of science put in different lights. They thus obtain in his mind their natural order — an order independent of all arbitrary and accidental associations.

The *Reductio ad Absurdum*, which to the mere beginner is a 'stumbling block' and 'foolishness,' is deferred until sufficient strength has been acquired in the use of the direct method. A happy effect of this arrangement is a new incentive in the novelty and beauty of the indirect, as well as the clear demonstration, within several sentences, of theorems which have hitherto occupied from a half to an entire page.

The subject of Constructions is strongly emphasized in artistic execution and disciplinary results. Nor is its relation to the practical arts neglected. Such allusions cannot fail to render the study both more interesting and more useful.

There has been constant care to simplify both language and subject matter, yet never at the sacrifice of accuracy or rigor. In presence of the many attempts to abbreviate, it should be remembered that real brevity is to be obtained, not by sparing words to the author, but by sparing time and labor to the reader.

No theorem is stated in connection with its converse. Separate topics are made separate lessons. Notation has been greatly reduced. Where it has been customary to use many letters in designating the diagram, I have, where possible, used but one, it being easier for the mind to grasp a single point than to embrace several, or to trace a broken line.

The diagrams are large and distinct, and are so placed as to enable the student readily to refer to them at every step. In the few cases where it becomes necessary to turn the page in reading a demonstration, the diagram is repeated.

In general, great pains have been taken to secure convenience, elegance, and distinctness of form. The advantages, for instance, of a full, clear, free typography, are not liable to be overestimated. The force otherwise expended in disengaging the idea from its dress, and one idea from another, is thus at once concentrated upon the *thing* and its *relations.* It is not an accident, but a law, that in proportion as there is attention to the sign there is inattention to the thing signified.

From first to last I have kept in view the student of average ability, and have written in the continual presence of those difficulties which I have had to meet as an instructor. As nearly as possible I have written as I have found myself obliged to teach, and thus hope to have furnished a thoroughly intelligible companion text. Tastes and habits are determined in the privacy of preparation. If perplexities are experienced here which must await the explanations of the class-room, the effect cannot be other than harmful.

So many writers have been consulted that an acknowledgment of indebtedness would be little else than the catalogue of a library. It will be sufficient to observe that the volume would have been different, as respects its matter, its spirit, and its form, had not the able treatises of Todhunter, Chauvenet, Olney and Schuyler appeared before it, as well as the several publications of the English Association for the Improvement of Geometrical Teaching; but that its differential features from these and others now current abundantly justify its existence, is confidently believed. Every man builds upon his predecessors, and all systems are but approximations to the ideal which no one has ever actually reached.

It remains only to express my sense of obligation to my able successor in the chair of mathematics, Dr. E. Fraunfelter, who has kindly read the proof-sheets and furnished me with many valuable suggestions.

THE AUTHOR.

Columbus, Ohio, June 15, 1883.

PART I.

PLANE GEOMETRY.

CHAPTER I.

THE SUBJECT-MATTER.

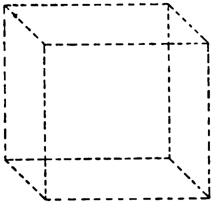
EVERY object that we can touch or see, occupies a portion of space, is extended, and has some outline.

For example, conceive this book to be suspended in mid-air: it has a certain length, width, and thickness, and a particular shape, and is distinguished by a particular form.

If, now, the book be removed, the space which it filled will still remain; and we can imagine it to have, as it does, the same form as the book had. It is not a material thing, like the book, but immaterial, yet having the same shape and the same extent.

This portion of space is called a *Geometric* solid, in distinction from the book itself, which is a *Physical* solid. A Geometric solid, then, is the *space* occupied by a physical solid, and has: first, three dimensions,— length, breadth, and thickness; second, a definite form, its form being the same as that of the physical solid.

Now, let the annexed diagram represent a geometric solid, or portion of space. The boundaries which separate this from all surrounding space are called **Surfaces**. As the boundary is not a part of the things separated, which are nevertheless in immediate contact, it is plain that a surface has no *thickness*, only length and breadth.



It is clear, also, that the bounding surfaces are themselves limited, or bounded. The limits or edges of surfaces are called **Lines**,—which being no part of the surface, but merely its *limits*, have length only.

The limits, or extremities, of lines are called **Points**. As they mark merely the limits or *ends* of lines, and thus are no part of the lines themselves, it is evident that they have no length,—they are not long nor wide nor thick.

Points may be considered without reference to lines, merely as positions in space. Lines may be considered without reference to surfaces, merely as extension in one dimension — length.

Surfaces may be considered without reference to geometric solids, merely as extension in two dimensions — length and breadth.

In reference to their extent, lines, surfaces, and solids are called magnitudes; in reference to their shape, figures. Figures formed by straight lines are called *rectilinear*.

It is thus seen that Geometry deals, not with material, but with immaterial, things. It treats of physical bodies by considering *the space which they fill;* and the student should distinctly realize and remember that, while points, lines, surfaces and geometric solids can be indicated and represented only in a material way, yet the points, lines, surfaces and solids of Geometry are purely ideal, though we *know* them to exist, and endeavor to make them apparent to the eye. For instance, while any line he may draw will have some breadth, and even thickness, or be otherwise imperfect, and, so far, fail to represent the *true* line, yet his reasoning should proceed on the supposition that the drawing is perfect — that the line drawn has *no* breadth.

DEFINITIONS.

These considerations make clear the following definitions:

1. A **Point** is that which has position, but no extension.

2. A **Line** is that which has extension in one dimension—*length.*

3. A **Surface** is that which has extension in two dimensions—*length* and *breadth.*

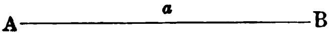
4. A Geometric **Solid**, or **Volume**, is that which has extension in three dimensions—*length, breadth,* and *thickness.*

5. **Geometry**, which treats of the properties and relations of Points, Lines, Surfaces, and Volumes, may be defined as *The Science of Position, Extension, and Form.*

LINES.

A *Straight Line,* or a **Line** simply, is one whose direction does not change.

It is designated by letters placed at two of its points, or by a small *italic* letter placed near its middle. Thus, the line A B or the line *a.* We may assign to it either direction—from B to A, or from A to B.

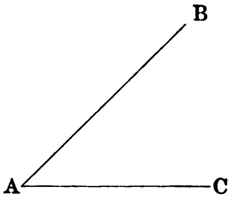


ANGLES.

When the legs of a pair of compasses are opened, they point in different directions, and are said to make an *angle* with each other. If, now, we imagine the legs to be replaced by geometrical lines, we shall have an angle as understood in Geometry.

1. An **Angle**, therefore, is the difference in direction of two lines which actually do meet, or would if sufficiently prolonged

Thus the opening between the lines A B and A C, at the point A, is an angle.

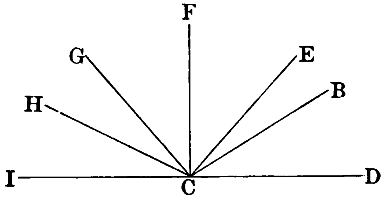


2. The lines which form the angle are called *Sides;* the point at which the sides meet is called the *Vertex.* Thus, A B and A C are the sides of the angle, and A the vertex.

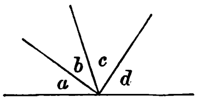
3. The degree of opening evidently remains the same, what-

ever be the *length* of the sides; that is, the magnitude, or size, of an angle depends upon the *position* of the sides, not their length.

4. If there is but one angle at a point, it may be read by one letter or by three, special care being taken to read the letter at the vertex *between* the other two. Thus, the above angle may be read, angle B A C, or C A B, or simply angle A.

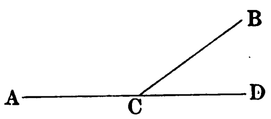


5. Any number of angles, however, may be formed at a given point, and so have a common vertex; in which case a particular angle must, in general, be designated and read by three letters; as, the angle B C D, or the angle G C F.

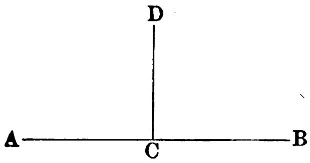


6. It is sometimes convenient to denote angles by small *italics* placed between the sides and near the vertex, in which case the single letter sufficiently indicates what angle is meant; as, the angles *a*, *b*, *c*, and *d*.

7. **Adjacent Angles** are those which have a common vertex and a common side between them. Thus, in the last figure, the angles *a* and *b* are adjacent; and, in the previous figure, the angles E C F and F C G, having the common vertex C and the common side C F, are adjacent; also, in the annexed figure, the angles A C B and B C D.

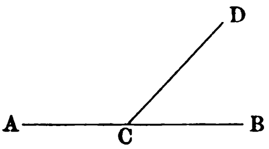


8. When one straight line meets another straight line so as to make the adjacent angles equal, each of these angles is a **Right Angle**, and the first line is said to be *perpendicular* to the second. Thus, if the two adjacent angles A C D and B C D, formed by the line C D meeting A B, are equal, each is a right angle, and D C is perpendicular to A B.

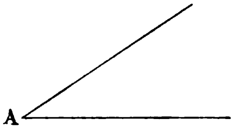


9. If one line meets another so as to make the adjacent angles *unequal*, each of these angles is called **Oblique**, in distinction from the right angle; as, A C D and D C B.

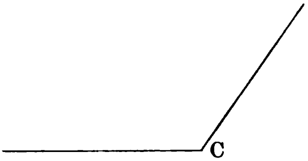
Oblique angles are of two classes: *acute* and *obtuse.*



10. An **Acute Angle** is one which is less than a right angle, as the angle A.



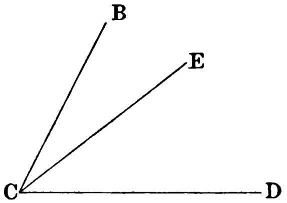
11. An **Obtuse Angle** is one which is greater than a right angle; as, the angle C, or the angle A C D above.



12. Angles, like other quantities, may be added to, or subtracted from, each other. Thus:

$$BCD = BCE + ECD,$$

and $BCE = BCD - ECD.$



13. The **Complement** of an angle is the difference between a right angle and the given angle.

14. The **Supplement** of an angle is the difference between two right angles and the given angle.

GENERAL TERMS.

1. A **Proposition** is something proposed or stated; as, 'Youth is thoughtless.'

2. A **Demonstration** is a course of reasoning by which the truth of a statement becomes evident. Thus:

All men are mortal;
Socrates was a man;
Therefore Socrates was mortal.

3. A **Theorem** is a truth requiring demonstration. Something is asserted, and the truth of it required to be proved.

4. A **Corollary** is a truth which follows, as a necessary consequence, from one or more preceding propositions.

5. A **Hypothesis** is a supposition assumed to be true in order to argue from it the truth or falsity of a proposition; and it is made either in the statement of the point in question, or in the course of the reasoning.

6. A **Scholium** is a remark upon some feature of a proposition.

AXIOMS.

An **Axiom** is a self-evident truth:

1. *Things which are equal to the same thing are equal to each other.*

2. *If the same operation be performed upon equals, the results will be equal.* Thus, if the same quantity be added to each of two equal quantities, the sums will be equal.

3. *If the same operation be performed upon unequals, the results will be unequal.* For instance, if the same quantity be added to each of two unequal quantities, it is plain that the results must be unequal.

4. *The whole is greater than any of its parts.*

5. *The whole is equal to the sum of all its parts.*

6. *All right angles are equal.*

7. *Only one straight line can be drawn from one point to another.*

8. *A straight line is the shortest distance between any two points.*

POSTULATES.

A **Postulate** is a self-evident possibility:

1. *A straight line can be drawn from any one point to any other.*

2. *A straight line may be prolonged to any distance.*

3. *A finite straight line can be bisected;* that is, divided into two equal parts.

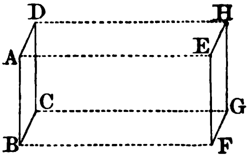
4. *An angle may be bisected.*

5. *An angle may be constructed equal to a given angle.*

6. *A straight line may be drawn perpendicular to a given straight line either from a point without the line or from a point in it.*

Definitions, axioms, and postulates form the basis of geometrical science.

EXERCISES.



1. Take a 'square' piece of paper, and suspending it in the position A B C D, move it to the position E F G H; then the form of the entire figure will represent the space through which the paper has passed, or the path which the paper has described. What kind of a solid is this path?

What kind of a solid is the paper itself?

One edge of the paper is A B, and A B F E represents its path in passing from the first position to the second. What is this path called? How, then, may a surface be defined?

What represents the path of A D? of D C? of B C?

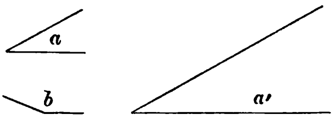
The corners of the paper are A, B, C, and D. The path of A is A E; what is it called? The path of B is what? of C? of D?

Hence a line may be considered as the path of what?

2. Does a line in motion necessarily generate a surface?

3. Why are two letters sufficient to read a straight line?

4. Which is the greater angle, *a* or *a'*? *a* or *b*?

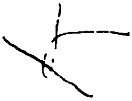


5. We call a string or telegraph wire a *line*. What is it, strictly speaking?

6. What kind of a point, really, is that made on paper with the end of a pencil?

7. How many straight lines can be drawn through the north and south poles of the earth?

8. How many straight lines can be drawn through 15 points, each line connecting two points?

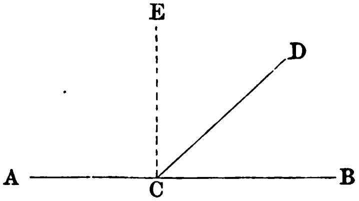


CHAPTER II.

RECTILINEAR FIGURES.

THEOREM I.

If one straight line meet another straight line, the sum of the adjacent angles will be equal to two right angles.



Let the line CD meet the line AB at the point C; then will $ACD + BCD =$ *two right angles.*

For, let CE be drawn perpendicular to AB at C (*Post. 6*), and let R denote a right angle. By definition, each of the angles ACE and ECB is a right angle; and therefore their sum equals two right angles; that is,

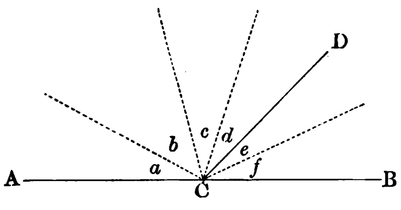
$$ACE + ECB = 2\,R.$$

Now it is plain that ACD and BCD together fill exactly the same space as ACE and ECB; hence,

$$\begin{aligned} ACD + BCD &= ACE + ECB \\ &= 2\,R. \end{aligned}$$

Therefore, *if one straight line meet another straight line, the sum of the adjacent angles will be equal to two right angles.*

Cor.—*The sum of all the angles formed at a point on the same side of a straight line is equal to two right angles.*



For,

(1) $a + b + c + d + e + f$ $=$ *all the space on the upper side of* $AB = ACD + DCB$.

But,

(2) $ACD + DCB = 2R$, as just proved;

$\therefore$ (3) $a + b + c + d + e + f = 2R$.

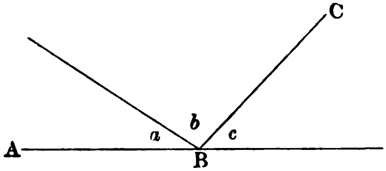
QUERIES.

1. On what two axioms is (1) founded?

2. By what axiom is (3) deduced from (1) and (2)? What is the 'same thing'? What are the 'equal things'?

3. In the proof of the theorem, what axiom is involved in the second equation?

EXERCISES.



1. If an angle is a right angle, what is its complement?

2. If an angle is a right angle, what is its supplement?

3. If an angle is ⅔ of a right angle, what is its complement?

4. If an angle is ⅞ of a right angle, what is its supplement?

5. If a right angle = 90 degrees (expressed 90°), and $c = 60°$, $ABC =$ what?

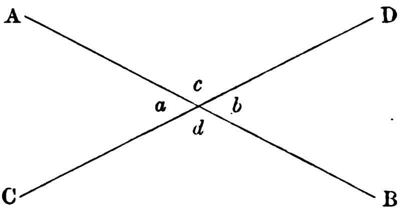
6. If $c = 50°$, and $a = 45°$, $b =$ what?

Note.—The student must not think he is bound to use the precise words, or follow in every particular, the precise arrangement of the author. Let him cultivate self-reliance and self-assertion,—the habit of original thought and original expression. To this end he is advised, in his beginning, after having studied a demonstration, to draw the figure a second time, using different letters; or a third time, using *no* letters. Let his sole aim be to discover the *ideas*, regardless of the expression; to distinguish, each by itself, the different steps in a demonstration; to ascertain and accurately to note the connection of one step with another, and how each succeeding step leads to the desired result, namely, the proof of his theorem. Lastly, these different steps should be separately fixed in the memory.

DEFINITIONS.

If two straight lines *cut*, or intersect, each other, four angles are formed about the point of intersection, which are distinguished as follows:

1. Those which lie on the same side of one of the lines and on opposite sides of the other are called **Adjacent Angles.** Thus, *a* and *c*, lying on the same side of D C and on opposite sides of A B, are adjacent; likewise, *a* and *d*, on the same side of A B and on opposite sides of D C, are adjacent.

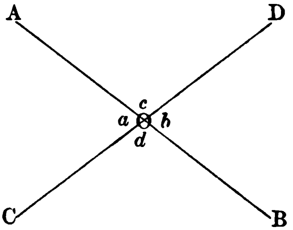


2. Those which lie on opposite sides of both lines are called **Vertical Angles.** Thus, a and b, lying on opposite sides of A B and on opposite sides of C D, are vertical; likewise, c and d.

Note.—Let the student see to it that he is able to give, in every instance, not only the number of the reference, but the reference itself, and that he understands clearly its use or application. It is all-important to his successful and pleasant study of Geometry that he should *start* well.

THEOREM II.

If two straight lines intersect, the vertical angles will be equal.



Let the two lines A B and C D intersect at O;

then $a = b$,

and $c = d$.

For, $a + c = 2\text{ R}$ - - - - - - - - - - - - - *Th. I.*

Also, $b + c = 2\text{ R}$ - - - - - - - - - - - - *Th. I.*

$\therefore$ $a + c = b + c$ - - - - - - - - - - - - *Ax. 1.*

$\therefore$ $a = b$ - - - - - - - - - - - - - - *Ax. 2.*

Again, $a + d = 2\text{ R}$ - - - - - - - - - - - - - *Th. I.*

Also, $a + c = 2\text{ R}$ - - - - - - - - - - - - *Th. I.*

$\therefore$ $a + c = a + d$ - - - - - - - - - - - - *Ax. 1.*

$\therefore$ $c = d$ - - - - - - - - - - - - - - *Ax. 2.*

Therefore, *if two straight lines intersect, the vertical angles will be equal.*

Cor. I. *If two straight lines intersect, the sum of the four angles formed about the point of intersection will be equal to four right angles.*

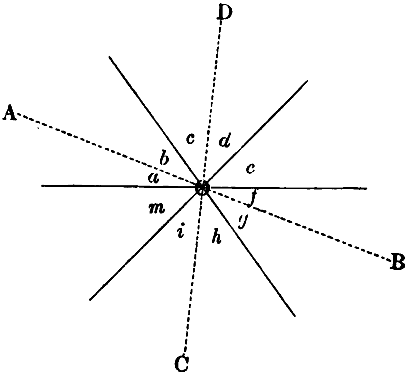
For, (1) $a + c = 2\text{ R}$ - - - - - - - - - - - - *Th. I.*

(2) $b + d = 2\text{ R}$ - - - - - - - - - - - *Th. I.*

(1) + (2) = (3) $a + c + b + d = 4\text{ R}$ - - - - - - - - *Ax. 2.*

Cor. II.—*The sum of all the angles that can be formed about a point is equal to four right angles.*

For, $a + b + c + d + e + f + g + h + i + m$
= *whole amount of space about O*
$= AOD + DOB + BOC + COA$
$= 4$ R - - - - *Cor. I.*



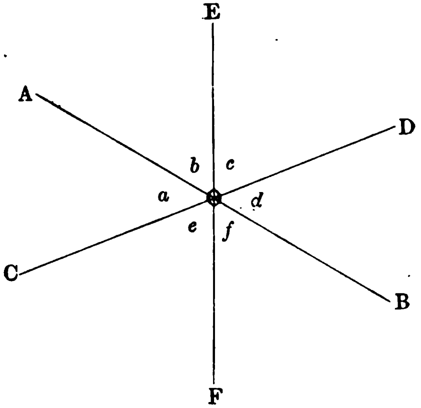
EXERCISES.

1. If $c = 120°$, $d =$ what?

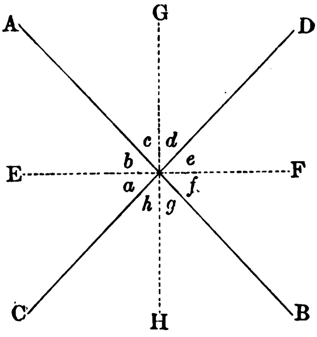
2. If a is $\frac{1}{3}$ of a right angle, what are the respective values of c, b, d?

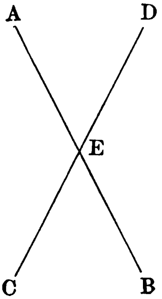
3. Prove, from the annexed diagram, that the straight line which bisects an angle bisects its vertical angle.

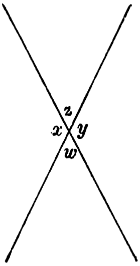
Suggestion.— Suppose the angle A O D to be bisected.



4. Prove, from the annexed diagram, that the bisectors of two pairs of vertical angles are perpendicular to each other.







5. Demonstrate Theorem II from each of the accompanying diagrams.

6. In how many points can two lines intersect? (*See Ax.* 7.)

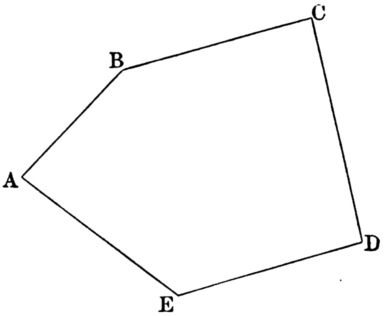
Three lines?

7. If five of six angles formed in a plane about a point are 12°, 30°, 48°, 75° and 148°, how many degrees are there in the sixth angle?

DEFINITIONS.

1. A **Plane Surface**, or simply a **Plane**, is a surface such that a straight line which joins any two of its points will lie wholly in the surface. Thus, it is evident that the straight line joining the points C and D will lie wholly in the surface of the paper — no part of it will pass through or *pierce* the paper.

2. A **Plane Polygon** is a portion of a plane surface bounded by straight lines. Thus, if any portion of the surface of this page be enclosed by straight lines, the resulting figure will be a plane polygon; as, A B C D E.



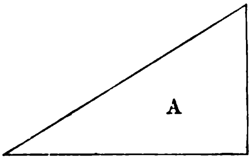
3. The bounding lines are called *sides* of the polygon. Thus, A B, B C, C D, D E, A E, are called *sides* of the polygon A B C D E.

4. The points in which the sides meet are called *vertices* of the polygon; as, A, B, C, etc.

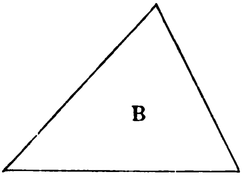
5. A **Plane Triangle** is a portion of a plane surface bounded by three straight lines; that is, a plane triangle is a polygon of three sides.

6. Triangles are classified, with reference to their angles, as *right*, and *oblique* { *acute*, *obtuse*.

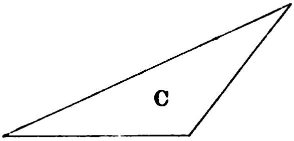
1st. A **Right Triangle** is one which has one right angle, as A.



2d. An **Acute Triangle** is one whose angles are all acute—each less than a right angle, as B.

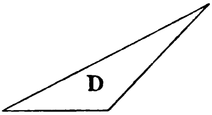


3d. An **Obtuse Triangle** is one which has one obtuse angle—an angle greater than a right angle, as C.

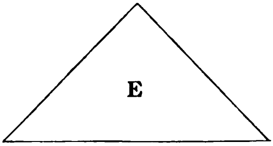


7. Triangles are classified, with reference to their sides, as *scalene*, *isosceles* and *equilateral*.

1st. A **Scalene Triangle** is one having no two sides equal, as D.

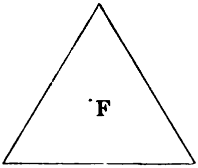


2d. An **Isosceles Triangle** is one having only two sides equal, as E.



3d. An **Equilateral Triangle** is one having all of its sides equal, as F.

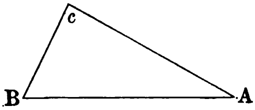
8. An **Equiangular Triangle** is one having all of its angles equal.



9. Polygons, and hence triangles, are *mutually* equilateral when their sides, taken in the same order, are equal each to each.

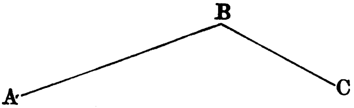
They are mutually equiangular when their angles, taken in the same order, are equal each to each.

10. The **Hypotenuse** of a right triangle is the side opposite the right angle.



11. The **Base** of a triangle is the side on which it is made to stand as, A B.

12. The angle opposite the base is called the **Vertical Angle**, as *c*.

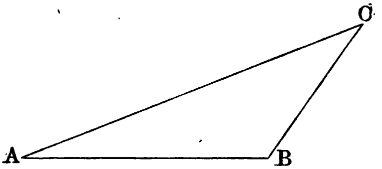


13. A **Broken Line** is a line consisting of two or more straight lines, as A B C, consisting of the lines A B and B C.

Plane Geometry, therefore, may be defined as the science which treats of plane figures—figures whose points are all in the same plane.

THEOREM III.

Any side of a triangle is less than the sum of the other two, and greater than their difference.



Let A B C be any triangle; then:

First: Any side will be less than the sum of the other two.

For, the direct distance from A to C, measured on the straight line A C, is less than the distance between the same points measured on the broken line A B C - - - - *Ax. 8.*

Hence,

(1) $AC < AB + BC.$

Likewise, (2) $AB < AC + AB.$

and (3) $BC < AC + AB.$

Second: Any side will be greater than the difference between the other two.

For, subtracting A B from each member of (1), for instance, we have $AC - AB < BC$;
that is, $BC > AC - AB$.

Therefore, *any side of a triangle is less than the sum of the other two, and greater than their difference.*

Note.—In the statement of any theorem, that which is *given* or *assumed*, in order to prove something else, is the Hypothesis, though the conditional 'if' may not be expressed. Thus in the preceding Theorem, the Hypothesis is the given triangle; that is, *if we have a triangle, or if a figure is a triangle.* The Conclusion is the statement whose truth is to be demonstrated: *any side will be less than the sum of the other two and greater than their difference.*

EXERCISES.

1. Can you draw a straight line on the surface of a ball?
2. Can a triangular field have one side as long as both the others?
3. Can it have one side less than, or equal to, the difference of the others?
4. Can a triangular lot have one side 40 rods, another 15 rods, and the third 20 rods, long?
5. Can two polygons, neither of which is equilateral, be mutually equilateral?
6. Can two polygons, neither of which is equiangular, be mutually equiangular?

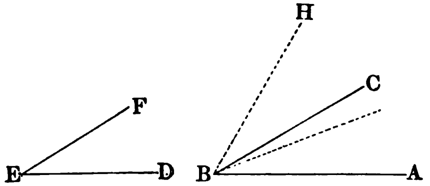
METHOD OF SUPERPOSITION.

Two geometrical magnitudes are equal in all their parts when they can be so adjusted, one to the other, as to coincide point for point. Such magnitudes are also said to be *congruent.*

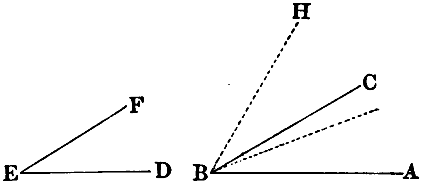
Thus, two angles are congruent, or equal throughout, if they can be so placed that their vertices coincide in position, and their sides in direction.

This assumes that a line may be moved from one place to another, without altering its length; and that an angle may be taken up and folded over, without altering the divergence of its sides.

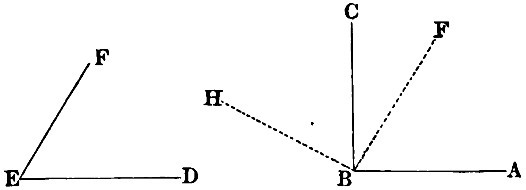
Thus, let angle D E F be applied to angle A B C, so that the side D E falls upon A B, and the vertex E falls upon the vertex B:



15.



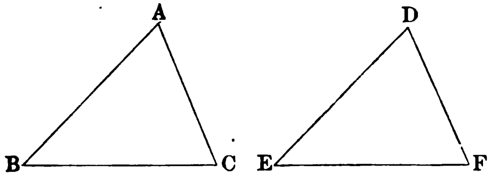
then, if E F falls upon B C, angle D E F = angle A B C; if E F falls *between* A B and B C, angle D E F < angle A B C; if E F falls *without* B C, as in the direction B H, angle D E F > angle A B C.



Again, let angle D E F be applied to A B C, so that the vertex E falls on the vertex B, and E D takes the direction B A; then E F will take the direction B F, and C B F = A B C − D E F. Or, let D E F be so applied to A B C that E shall fall on B, and E D take the direction B C; then E F will take the direction B H, and A B H = A B C + D E F.

THEOREM IV.

If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the triangles will be equal in all their parts.

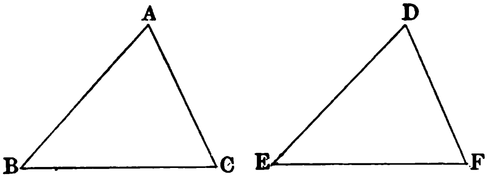


Let A B C and D E F be two triangles, having the side A B equal to the side D E, and the side A C equal to the side D F, and the included angle A equal to the included angle D; then will the triangles be equal in all their parts. For, let the triangle A B C be applied to the triangle D E F so that the angle A shall coincide with the equal angle D, the side A B coinciding with the equal side D E, and the side A C coinciding with the equal side D F: then, since B falls upon E, and C upon F, the base B C must coincide with the base E F; for between two points only one straight line can be drawn, connecting them (*Ax.* 7); hence, the triangles coincide throughout.

Therefore, *if two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the triangles will be equal in all their parts.*

THEOREM V.

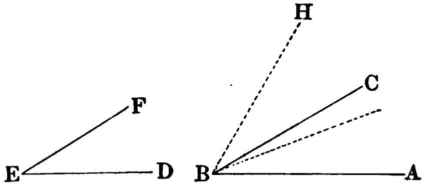
If two triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, the triangles will be equal in all their parts.



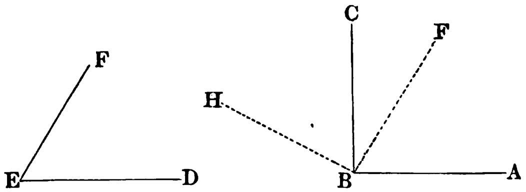
Let A B C and D E F be two triangles, having the angle B equal to the angle E, the angle C to the angle F, and the included side B C equal to the included side E F; then will the triangles be equal in all their parts.

For, apply the triangle A B C to the triangle D E F so that the base B C shall coincide with its equal, E F; then, since the angle B is equal to the angle E, the side B A must fall upon E D, and the point A will lie somewhere on E D; also, since the angle C is equal to the angle F, the side C A must fall upon F D, and the point A will lie somewhere on F D. Now, if the point A is at once on E D and F D, it must be at their point of intersection, since they have no other point in common; hence, the triangles coincide throughout.

Therefore, *if two triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, the triangles will be equal in all their parts.*



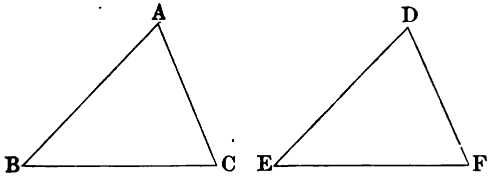
then, if E F falls upon B C, angle $DEF =$ angle ABC; if E F falls *between* A B and B C, angle $DEF <$ angle ABC; if E F falls *without* B C, as in the direction B H, angle $DEF >$ angle ABC.



Again, let angle D E F be applied to A B C, so that the vertex E falls on the vertex B, and E D takes the direction B A; then E F will take the direction B F, and $CBF = ABC - DEF$. Or, let D E F be so applied to A B C that E shall fall on B, and E D take the direction B C; then E F will take the direction B H, and $ABH = ABC + DEF$.

THEOREM IV.

If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the triangles will be equal in all their parts.

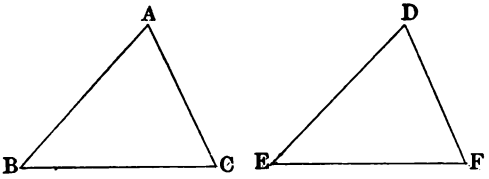


Let A B C and D E F be two triangles, having the side A B equal to the side D E, and the side A C equal to the side D F, and the included angle A equal to the included angle D; then will the triangles be equal in all their parts. For, let the triangle A B C be applied to the triangle D E F so that the angle A shall coincide with the equal angle D, the side A B coinciding with the equal side D E, and the side A C coinciding with the equal side D F: then, since B falls upon E, and C upon F, the base B C must coincide with the base E F; for between two points only one straight line can be drawn, connecting them (*Ax.* 7); hence, the triangles coincide throughout.

Therefore, *if two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the triangles will be equal in all their parts.*

THEOREM V.

If two triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, the triangles will be equal in all their parts.



Let A B C and D E F be two triangles, having the angle B equal to the angle E, the angle C to the angle F, and the included side B C equal to the included side E F; then will the triangles be equal in all their parts.

For, apply the triangle A B C to the triangle D E F so that the base B C shall coincide with its equal, E F; then, since the angle B is equal to the angle E, the side B A must fall upon E D, and the point A will lie somewhere on E D; also, since the angle C is equal to the angle F, the side C A must fall upon F D, and the point A will lie somewhere on F D. Now, if the point A is at once on E D and F D, it must be at their point of intersection, since they have no other point in common; hence, the triangles coincide throughout.

Therefore, *if two triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, the triangles will be equal in all their parts.*

Scholium.—It will aid the student's memory to note carefully the distinction between the above Theorem and the preceding one; in the first the Hypothesis is that two *sides* and the included *angle* of the one are equal to two *sides* and the included *angle* of the other; in the second, the Hypothesis is that two *angles* and the included *side* of the one are equal to two *angles* and the included *side* of the other. In the first, there are given two *sides* and the included angle; in the second, two *angles* and the included side.

QUERIES.

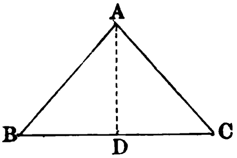
1. How much, or what part, of the last Theorem is *Conclusion?*
2. What part of Theorem I is *Hypothesis?* Of Theorem II?
3. What part of each is *Conclusion?*
4. Which is the precise thing to be *proved*, the Hypothesis or the Conclusion?
5. In Theorem VI, what is *given*, and what is required to be *proved?*

EXERCISES.

1. State the fact that 'Socrates was mortal,' in the form of a theorem, with hypothesis and conclusion.
2. Of two supplementary angles, the greater is twice the less; find what fraction the less is of four right angles.

THEOREM VI.

In an isosceles triangle, the angles opposite the equal sides are equal.

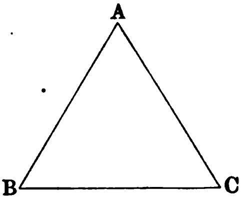


Let A B C be an isosceles triangle, having the side A B equal to the side A C; then will the angle B, opposite A C, be equal to the angle C, opposite A B.

For, draw A D, bisecting the angle B A C (*Post. 4*). The given triangle is divided into the triangles A B D and A D C. A B, of the triangle A B D, is equal to A C, of the triangle A D C, by hypothesis; A D is common; and the included angle B A D, of the first, is equal to the included angle D A C, of the second, *by construction;* hence, the two triangles are equal in all their parts (*Th. IV*); and, consequently, the angle B is equal to the angle C.

Therefore, *in an isosceles triangle, the angles opposite the equal sides are equal.*

Cor. I.—*An equilateral triangle is equiangular.*



For, if A B equals A C, then, by the Theorem just proved, we have (1) B = C; and, if A B = B C, we have, for the same reason, (2) A = C: combining (1) and (2) we have

(3) A = B - - - - - - - *Ax. 1.*

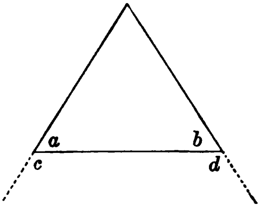
∴ (4) A = B = C.

Cor. II.—*The line which bisects the vertical angle of an isosceles triangle, is perpendicular to the base at its middle point.*

Cor. III.—*The perpendicular from the vertex of an isosceles triangle to the base, bisects the base and the vertical angle.*

EXERCISES.

1. If $a=\frac{2}{3}R$, what is the value of b, the triangle being isosceles?



2. If the equal sides of an isosceles triangle be produced, prove that the exterior angles formed with the base are equal; that is, $c=d$.

3. Draw three triangles having a common side; three having a common angle.

4. On a given line as base, how many isosceles triangles can be constructed?

Note.—The figure which is drawn to conform merely to the Hypothesis, and is only the expression of it in pictorial form, is said to be *given;* if lines are added afterward, for purposes of demonstration, the lines so drawn are called the **Construction,** as A D above.

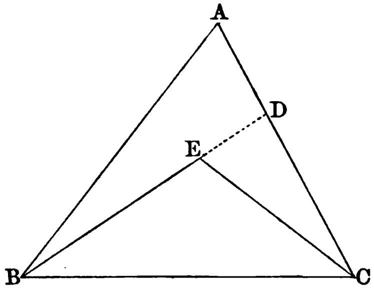
The student, then, will understand that his work consists of five principal parts:

1. To give a *general* statement of his Theorem.

2. To give a *particular* statement of it with reference to the figure.
3. To give the *construction*, if there be any.
4. The *demonstration*.
5. The *conclusion*.

THEOREM VII.

If from a point within a triangle two straight lines be drawn to the extremities of any side, their sum will be less than that of the other two sides.[1]



Let E be any point within the triangle A B C; and from this point let B E and E C be drawn to the extremities of any side, as B C: then

$BE + EC < AB + AC$.[2]

For, produce one of the lines, as B E, till it meets A C in D:[3] then in the triangle E D C,

$EC < ED + DC$. - - - *Th. III.*

∴ $BE + EC < BE + ED + DC$.

But $BE + ED = BD$. - - - - - - - - *Ax. 5.*

∴ $BE + EC < BD + DC$.

Again, $BD < AB + AD$. - - - *Th. III.*

∴ $BD + DC < AB + AD + DC$.

But $AD + DC = AC$. - - - - - - - - *Ax. 5.*

∴ $BD + DC < AB + AC$.

Now, $BE + EC < BD + DC$.

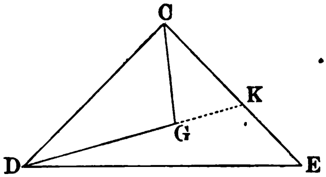
Much more then, $BE + EC < AB + AC$.

Therefore, *if from a point within a triangle two straight lines be drawn to the extremities of any side, their sum will be less than that of the other two sides.*

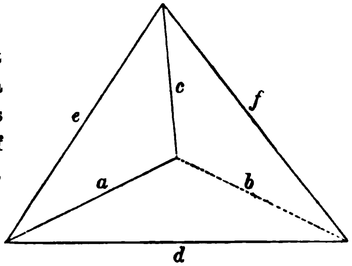
[1] General statement. [2] Particular statement. [3] Construction.

EXERCISES.

1. Demonstrate Theorem VII from the annexed diagram.



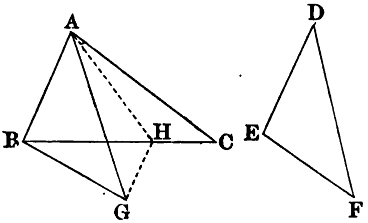
2. The sum of the three straight lines drawn from any point within a triangle to the three vertices, is less than the sum and greater than the half sum of the three sides of the triangle. (*See Ths. III and VII.*)



THEOREM VIII.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the included angle of the first greater than the included angle of the second, the third side of the first will be greater than the third side of the second.

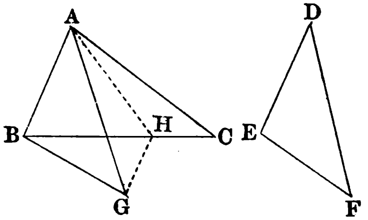
Let A B C and D E F be two triangles, having the side A B = D E, the side A C = D F, but the angle B A C > the angle E D F; then will the third side B C > E F.



For, apply the triangle D E F to the triangle A B C so that E D shall coincide with its equal B A; then, because the angle B A C > the angle E D F, the side D F will fall *within* the triangle A B C, and the triangle D E F will take the position A B G. Bisect the angle G A C by A H (*Post. 4*), and draw G H. In the triangles A H G and A H C, A G = A C, by hypothesis. A H is common, and included angle H A G = included angle H A C, by construction.

$\therefore \quad GH = HC$ - - - - - - - - *Th. IV.*

$\therefore \quad GH + BH = HC + BH$ - - - - - - - *Ax. 2.*



But $HC + BH = BC$ - *Ax. 5.*

$\therefore \ GH + BH = BC$ - *Ax. 1.*

But $GH + BH > BG$ *Th. III.*

$\therefore \ BC > BG.$

But $EF = BG.$

$\therefore \ BC > EF.$

Therefore, *if two triangles have two sides of the one equal to two sides of the other, each to each, but the included angle of the first greater than the included angle of the second, the third side of the first will be greater than the third side of the second.*

QUERIES.

1. What is the general statement of the above Theorem?

2. Where does the particular statement begin and end?

3. What is the construction? that is, what additional lines were drawn for purposes of demonstration?

4. In the general statement of the Theorem, with what word does the Hypothesis end?

5. What is the first step in the demonstration proper?—*To apply the triangle D E F to the triangle A B C.*

6. What is the second step?—*To bisect the angle G A C, and draw G H.*

7. The third step?—*To prove the triangles A H G and A H C equal.*

8. To what end, for what purpose, do you prove these triangles equal?—*To show that G H = H C.*

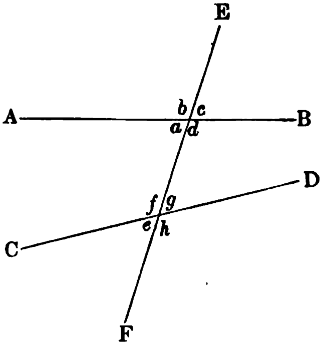
9. Why prove that G H = H C?—*To show that B H + G H = B C.*

10. Why prove this?—*To show that B C > B G.*

Note.—Whenever he has made a statement, or established a point, let the student as an aid to his memory, in recalling the successive steps, ask the question, *What of it?* What *use* am I to make of it?

DEFINITIONS.

1. When a straight line intersects two other straight lines it is called their **Secant.** Thus, E F is a secant of the two lines A B and C D.



2. The eight angles formed about the points of intersection are distinguished thus:

(1) The four angles *between* the two lines are called **Interior**; as, *a*, *d*, *f*, *g*.

(2) The other four are called **Exterior**; as, *b*, *c*, *e*, *h*.

(3) **Interior Angles on the Same Side** are those which lie *within* the two lines, and on the *same* side of the secant; as, *a* and *f*, or *d* and *g*.

(4) **Exterior Angles on the Same Side** are those which lie *without* the two lines, and on the same side of the secant; as, *b* and *e*, or *c* and *h*.

(5) **Alternate Interior Angles** are those which lie *within* the two lines, and on *different* sides of the secant, but not adjacent; as, *a* and *g*, or *f* and *d*.

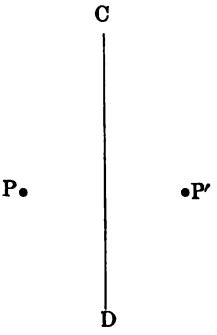
(6) **Alternate Exterior Angles** are those which lie *without* the two lines, and on different sides of the secant, but not adjacent; as, *b* and *h*, or *c* and *e*.

(7) **Opposite Exterior** *and* **Interior Angles** are those which lie on the same side of the secant, the one *within* and the other *without* the two lines, but not adjacent; as, *c* and *g*, or *h* and *d*, or *b* and *f*.

3. **Parallel Lines** are those which have the same direction, and cannot meet, however far either way both may be produced.

Cor.—*Two parallel lines are everywhere equally distant.*

4. A **Locus**[1] is the line[2] or surface, all of whose points have a common property which does not belong to any other point.



Thus, if every point in C D is equally distant from P and P′, and if no point outside of C D *is* equally distant from P and P′, then the line C D is the *locus*, or *place*, of all the points equally distant from the two fixed points P and P′; that is, all the points which satisfy the condition,—equality of distance from P and P′,—are in C D, and *nowhere else.* Or thus: C D is the locus, or place, of a point which moves according to a given law, namely, so as to be always equally distant from P and P′.

EXERCISES.

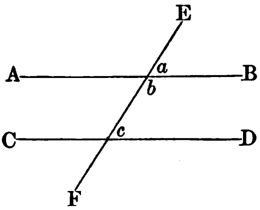
1. In the plane of a line, what is the locus of all the points 3 inches distant from that line?

2. Does this locus consist of more than one line?

3. How shall a straight line intersect two parallel lines in order that all the angles may be equal?

THEOREM IX.

If a straight line cut two other straight lines, making the sum of the interior angles on the same side equal to two right angles, the two lines cut will be parallel.



Let the straight line E F cut the two straight lines A B and C D, making the sum of the interior angles on the same side equal to two right angles; then will A B and C D be parallel.

---

[1] The Latin for *place.*

[2] The locus *may* consist of a plurality of lines; but it is important to think of them as detached *parts* of a *whole*, or as members of one and the same locus.

For, (1) $a + b = 2\,R$ - - - - - - - - - - - - *Th. I.*

And (2) $b + c = 2\,R$ by hypothesis:

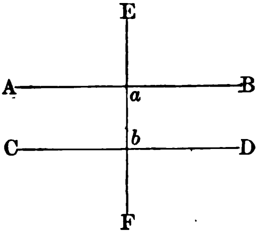
hence, (3) $a + b = b + c$ - - - - - - - - - - - - *Ax. 1.*

$\therefore$ (4) $a = c$ - - - - - - - - - - - - - *Ax. 2.*

Now, since E F remains fixed in position, if the angle *a* equals the angle *c*, the line A B must depart, or diverge, from E F just the same as C D does; that is, the two lines must have the same direction; if they have the same direction, then, by definition, they are parallel.

Therefore, *if a straight line cut two other straight lines, making the sum of the interior angles on the same side equal to two right angles, the two lines cut will be parallel.*

Cor.—*If two lines are perpendicular to a third, they are parallel.*

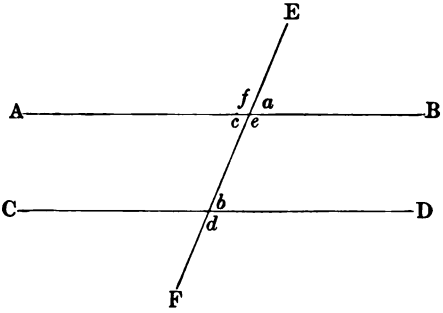


For, A B and C D being perpendicular to E F, each of the angles *a* and *b* is a right angle; and, consequently, their sum equals two right angles; hence, by the Theorem just proved, A B and C D are parallel.

EXERCISES.

Prove that, if one straight line cut two other straight lines, the two lines will be parallel, in the following cases:

1. When the alternate interior angles are equal, as *b* and *c*. (*See Theorem I or II.*)



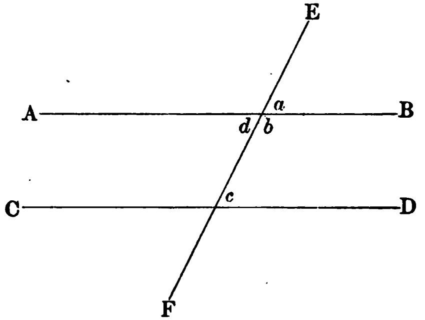
2. When the alternate exterior angles are equal, as *d* and *f*.

Suggestion.—(1) $a + f = 2\,R$, (2) $b + d = 2\,R$.

3. When the opposite exterior and interior angles are equal.

THEOREM X.

If a straight line cut two parallel straight lines, the opposite exterior and interior angles will be equal.



Let A B and C D be parallel, and let E F cut them; then will the opposite exterior and interior angles be equal.

For, if A B and C D are parallel, they must have, by definition, the same direction; if they have the same direction, then A B must diverge from E F just the same as C D does; that is, the amount of divergence, in each case is the same; hence, the angles are equal; that is, $a = c$.

Therefore, *If a straight line cut two parallel straight lines, the opposite exterior and interior angles will be equal.*

Cor. I.—*If a straight line cut two parallel lines, the sum of the interior angles on the same side equals two right angles.*

For, (1) $a + b = 2\text{ R}$ - - - - - - - - - - - - *Th. I.*

and, (2) $a = c$ - - - - - - - - - - - - - - *Th. X.*

$\therefore$ (3) $b + c = 2\text{ R}$.

Cor. II.—*If a line is perpendicular to one of two parallels, it is perpendicular to the other also.*

For, (1) $b + c = 2\text{ R}$ - - - - - - - - - - - - - *Cor. I.*

But, if E F is perpendicular to A B, for example, b is a right angle; that is (2) $b = \text{R}$.

Subtracting (2) from (1), we have

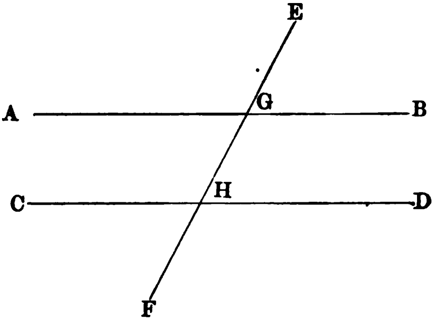
(3) $c = \text{R}$; but, if c is a right angle, E F must be perpendicular to C D.

EXERCISES.

1. Prove that, if a straight line cut two parallel lines, the alternate angles will be equal.

2. Prove that, if a straight line cut two parallel lines obliquely, the four acute angles formed about the points of intersection will be equal, and also the four obtuse angles.

3. Demonstrate Theorem X and its corollaries from the annexed diagram, not using the small italics nor the written equations.

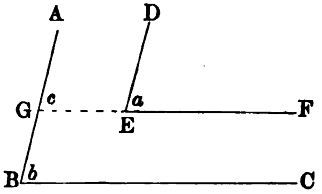


THEOREM XI.

Two angles which have their sides parallel and lying in the same or in opposite directions, are equal.

First:

Let the angles A B C and D E F have their sides parallel and lying in the same direction; then will they be equal.

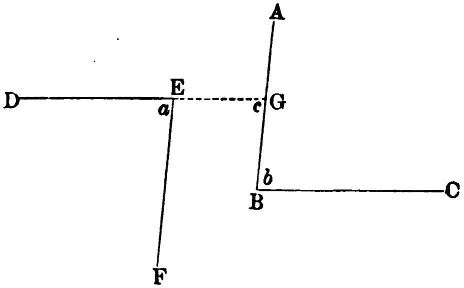


For, prolong E F to G; then, since A B and D E are parallel, the exterior angle a equals its opposite interior angle c; and, since B C and G F are parallel, the interior angle b is equal to its opposite exterior angle c; hence we have

(1) $a = c$, and (2) $b = c$; hence, (3) $a = b$.

Second:

Let B C be parallel to E D, but extending in the opposite direction; let B A be parallel to E F, but extending in the opposite direction; then will $a = b$.

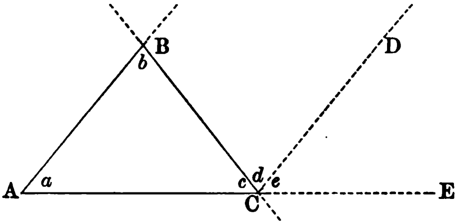


For, prolong D E to G; then, since B C and D G are parallel, and cut by A B, $b = c$ (*Th. X, Ex. 1*); but $a = c$ (*Th. X*); $\therefore$ $a = b$.

Therefore, *two angles which have their sides parallel and lying in the same or in opposite directions, are equal.*

THEOREM XII.

The sum of the angles of a triangle is equal to two right angles.



Let A B C be any triangle; then will the sum of its angles, a, b and c, be equal to two right angles.

For, produce the base A C to E, and draw C D parallel to A B; then, because A B and C D are parallel and cut by B C, the alternate angles b and d are equal; that is,

(1) $b = d$. Because the parallels

are cut by A E, (2) $a = e$.

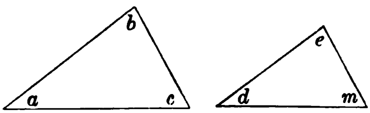
Also, (3) $c = c$.

(1) + (2) + (3) = (4) $b + a + c = d + e + c$.

But $d + e + c = 2$ R; hence, $b + a + c = 2$ R.

Therefore, *the sum of the angles of a triangle equals two right angles.*

Cor.—*If two triangles have two angles of the one equal to two angles of the other, the third angles will be equal.*



For,

(1) $a + b + c = 2$ R.

and (2) $d + e + m = 2$ R.

$\therefore$ (3) $a + b + c = d + e + m$.

If, now, $a = d$ and $b = e$,

then (4) $a + b = d + e$.

(3) − (4) = (5) $c = m$.

EXERCISES.

1. If $a = \frac{2}{3}$ R, and $b = \frac{5}{6}$ R, $c =$ what?

2. What is the value of each angle of an equilateral triangle?

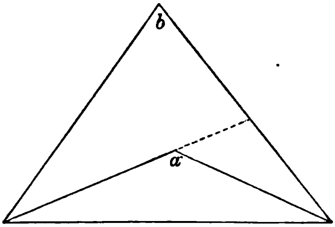
3. If one of two equal angles of a triangle is 45°, how many degrees in each of the others, and what is the kind of triangle?

4. Prove that the exterior angle, B C E, of a triangle is equal to the sum of the opposite interior angles a and b.

5. If the opposite interior angles of a triangle are 35° 25′, 68° 35′, what is the exterior angle?

6. If an interior angle of a triangle is 90°, what kind of a triangle is it?

7. The angle contained by the two lines drawn from any point within a triangle to the extremities of any side, is greater than the angle contained by the other two sides of the triangle. (*See Ex. 4.*)



THEOREM XIII.

In any triangle, the greater angle lies opposite the greater side.

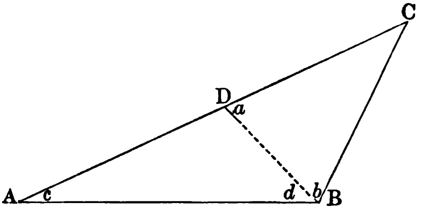
Let A B C be a triangle having

the side $AC > BC$;

then the angle $ABC > c$.

For, draw B D, making

$DC = BC$;



then $a = b$ - - - - - - - - - - - - - *Th. VI.*

But $a = c + d$ - - - - - - - *Th. XII, Ex. 4.*

$\therefore$ $b = c + d$. Adding d to each member,

$b + d = c + 2d$

$\therefore$ $b + d > c$

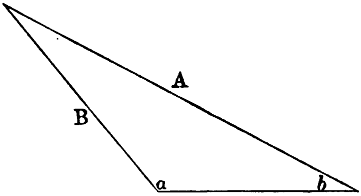
But $b + d = ABC$ - - - - - - - - - - - *Ax. 5.*

$\therefore$ $ABC > c$.

Therefore, *in any triangle, the greater angle lies opposite the greater side.*

EXERCISES.

1. Prove, from the annexed diagram, that the greater side lies opposite the greater angle.



Suggestion.—Suppose $a > b$;
then, if A is not $>$ B,
$A = B$, or $A < B$. But, if $A = B$, *etc.*

2. The vertical angle of an isosceles triangle is $\frac{2}{3}$ R, and the base is 6; required each of the other sides.

DEFINITIONS.

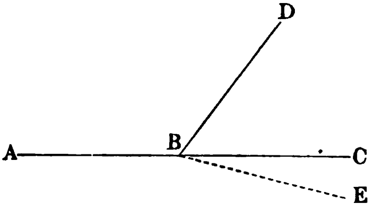
There are two distinct methods of demonstration — the *direct* and the *indirect*. The former, as we have seen, proves a proposition in either of two ways:

1. By applying one figure to another, as in Theorem III.
2. By the use of definitions, axioms, and principles previously established.

The latter — indirect, known also as the *reductio ad absurdum* — proves a proposition true by supposing it to be false, then showing that this supposition leads to a false conclusion — false either because it contradicts an axiom and is hence absurd, or because it contradicts a principle already proved or known to be true, hence involving the absurdity that one truth may contradict another. Since the conclusion is false, the supposition which logically led to it is false; but if the assumed proposition is false, its contradictory — the given proposition — is true, and is therefore demonstrated.

THEOREM XIV.

If a straight line meet two other straight lines at a common point, making the sum of the adjacent angles equal to two right angles, the two lines met will form one and the same straight line.



Let D B meet A B and C B at their common point B, making the sum of the adjacent angles, A B D and D B C, equal to two right angles; then will A B and C B form one and the same straight line.

For, if not, suppose A B and any other line than B C, as B E, to form the same straight line[1]:

Then will $ABD + DBE = 2R$ - - - - - - - - *Th. I.*

But, by hypothesis,

$$ABD + DBC = 2R;$$

hence, $$ABD + DBC = ABD + DBE:$$

Subtracting A B D from each of these equals, we have

$$DBC = DBE;$$[2]

that is, a part is equal to the whole, which is absurd.

Hence, the supposition that A B and C B do *not* form one and the same straight line, or which is the same, that A B and some other line than C B, as B E, do form the straight line, leads to the false conclusion that a part is equal to the whole; and, since the conclusion is false, this supposition is false: but, if it is false that A B and C B do not form one and the same straight line, it is true that they do.

Therefore, *if one straight line, etc.*

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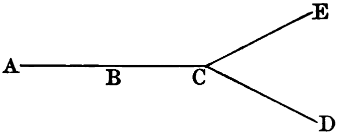
[1] Here we have supposed the proposition false, or, which is the same, that some other line than C B, as B E, is the prolongation of A B.

[2] This is the conclusion to which our 'supposition' has led us.

EXERCISES.

1. Prove, from the annexed diagram, by the *reductio ad absurdum*, that if two straight lines have two points in common, they coincide and form one and the same line.

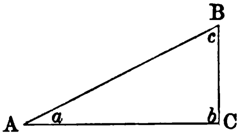


Suggestion.—Let A and B be the two common points. Between these points the lines must coincide (*Ax.* 7). But suppose they begin to separate at C; then one or the other must change its direction, which is contradictory to the definition of a straight line; hence, etc.

2. If the vertical or opposite angles formed by four straight lines, meeting at a common point, are equal, these four lines form but two straight lines.

THEOREM XV.

A triangle can have but one right angle.



Let A B C be a triangle, having the right angle b; then can it have no other right angle.

For, if it can have another, suppose a to be a right angle; then, since b is a right angle, we shall have

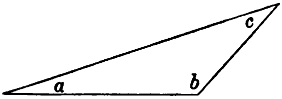
$$(1) \quad a + b = 2\,\mathrm{R};$$

but $$(2) \quad a + b + c = 2\,\mathrm{R} \text{ - - - } \textit{Th. XII.}$$

$(2) - (1) = (3) \quad c = 0$; that is, the third angle is 0, and the triangle has but two angles; which is impossible, since a triangle cannot exist without three angles. Hence, since the conclusion is false, the supposition that led to it is false; that is, it is false that the triangle can have more than one right angle; therefore it is true that it can have but one.

Therefore, *etc.*

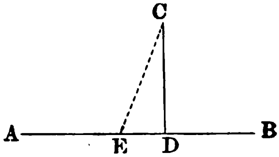
EXERCISE.



Prove that a triangle can have but one obtuse angle, using the *reductio ad absurdum.*

THEOREM XVI.

From a point without a line, only one perpendicular can be drawn to that line.



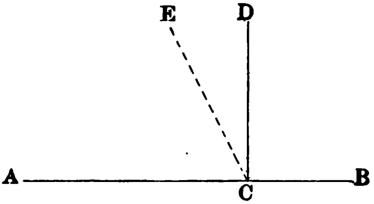
Let A B be a given line, and C a point without it; then can only one perpendicular, as C D, be drawn to A B from the point C.

For, suppose more than one can be drawn, and let C E be a second perpendicular from the given point to the given line: now, in the triangle C D E, the angle C D E is the right angle, since C D is perpendicular to A B by hypothesis; and *if* C E is also a perpendicular, the angle C E D must be a right angle, and the triangle would have two right angles, which is impossible (*Th. XV*). Hence, since the conclusion is false, the supposition which led to it is false; and if it is false that more than one perpendicular can be drawn to a line from a point without that line, it is true that only one can be.

Therefore, *etc.*

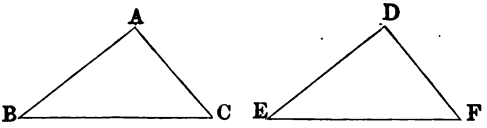
EXERCISE.

Prove, from the annexed diagram, by the *reductio ad absurdum*, that from a point in a line, only one perpendicular can be erected to the line.

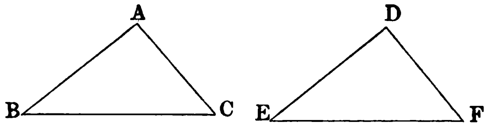


THEOREM XVII.

Two triangles which are mutually equilateral are mutually equiangular.



Let A B C and D E F be two triangles having A B = D E, A C = D F, and B C = E F: then will the angle A = D, B = E, and C = F.



For, taking the angles A and D, there are but three cases possible; namely,

$$A > D,\ A < D, \text{ or } A = D.$$

If $A > D$, $BC > EF$, which is contrary to the hypothesis, since $BC = EF$; $\therefore$ A is not $> D$. If $A < D$, $BC < EF$, which is contrary to the hypothesis; $\therefore$ A is not $< D$.

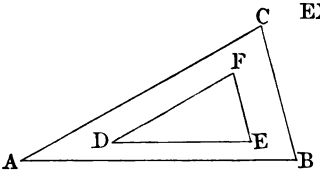
Hence, since A is neither $> D$ nor $< D$, A must $= D$. Then in the two triangles, we have two sides and the included angle of the one equal to two sides and the included angle of the other; hence, they are equal in all their parts; hence, $B = E$ and $C = F$.

Therefore, *etc.*

Scholium.—In triangles equal in all their parts, the equal sides lie opposite the equal angles, and the equal angles lie opposite the equal sides.

QUERIES.

1. If $A > D$, why is $BC > EF$?
2. If $A < D$, why is $BC < EF$?



EXERCISES.

1. Prove, by the *reductio ad absurdum*, that $B = E$, and $C = F$.

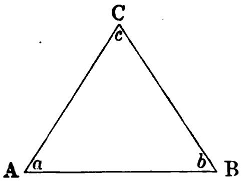
2. Are A B C and D E F mutually equiangular? *(See Th. XI.)*

Are they *equal?*

If, then, two triangles are mutually equiangular, does it follow that they are also mutually equilateral?

THEOREM XVIII.

If two angles of a triangle are equal, the sides opposite them are equal.



Let A B C be a triangle having the angles a and b equal; then will $BC = AC$.

For, if B C and A C are unequal, the angles lying opposite them are unequal, the greater angle lying opposite the greater side (*Th. XIII*), which is contrary to the hypothesis that a and b are equal; ∴ the conclusion is false; ∴ the supposition which led to this conclusion is false; that is, it is false that B C and A C are unequal; ∴ it is true that they are equal.

Therefore, *etc.*

Cor.—*Every equiangular triangle is equilateral.*

For, if $a = b$, then, as just proved, B C = A C;
and, if $a = c$, then, for the same reason, B C = A B;
∴ A B = A C = B C - - - - - - - - - - - - *Ax. 1.*

THEOREM XIX.

If from a point without a straight line, a perpendicular and oblique lines be drawn to that line, then:

1. *Oblique lines, which meet the given line at equal distances from the foot of the perpendicular, will be equal.*

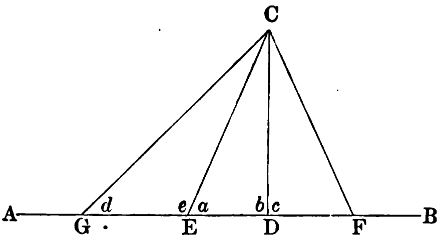
2. *Of two oblique lines meeting the given line at unequal distances from the foot of the perpendicular, the one meeting it at the greater distance will be the longer.*

3. *The perpendicular will be shorter than any oblique line.*

Let C be the given point, and A B the given line; C D the perpendicular; C E and C F oblique lines terminating at equal distances from D; C E and C G oblique lines terminating at unequal distances from D; then

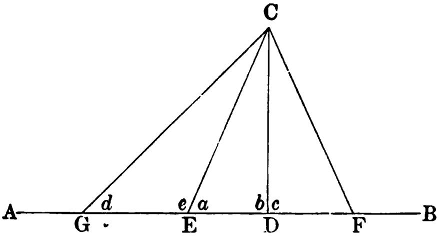
First: C E = C F.

For, in the triangles C D E and C D F, E D = D F by hypothesis, $b = c$, and C D is common: hence the triangles are equal throughout; ∴ C E = C F.



Second: Let G D > E D, then will G C > E C.

For, since b is a right angle, a is acute, and if a is acute, e must be obtuse; and, since a triangle can have but one obtuse angle, $e > d$; $\therefore$ G C > E C - - - - - - - *Th. XVIII, Ex.*



Third: b being a right angle, is $> a$; $\therefore$ C E > C D, or, which is the same thing, C D < C E; $\therefore$ C D must, for a still stronger reason, be < C G.

Therefore, *if from a point, etc.*

Cor.—*A perpendicular is the shortest distance from a point to a line.*

For, from the same point to the same line only *one* perpendicular can be drawn (*Th. XVI*); hence all other lines drawn from this point to the given line must be oblique; but by the Theorem just proved, the perpendicular is shorter than any oblique line.

QUERIES.

1. By what axiom does $b = c$?

2. On the supposition that b is a right angle, why must a be acute? (*See Th. XII.*)

3. If a is acute, why must e be obtuse? (*See Th. I.*)

4. By what Theorem are the triangles C D E and C D F shown to be equal?

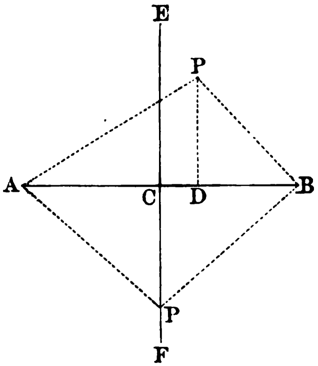
5. In the proof of 'First,' what is the important point?—*That the triangles, C D E and C D F, are equal.* In the proof of 'Second'?—*That e is an obtuse angle.* In the proof of 'Third'?—*That b is a right angle.*

THEOREM XX.

If a perpendicular be drawn to the middle point of a straight line, then:

1. *Any point of the perpendicular will be equally distant from the extremities of the line.*

2. *Any point outside of the perpendicular will be unequally distant from the extremities of the line.*



Let AB be the given line, C its middle point, EF the perpendicular, P any point in the perpendicular, and P' any point outside of it; then,

First: P is equally distant from A and B.

For, draw AP and BP; then, since by hypothesis $AC = BC$, $AP = BP$ (*Th. XIX*); that is, *any point of the perpendicular is equally distant from the extremities of the given line*, A and B.

Second: P' is unequally distant from A and B; that is, $AP' >$ or $< P'B$.

For, if from P' a perpendicular be let fall on AB, it must terminate either at the right or left of C; for, if it terminated at C, there would be two perpendiculars erected to a given line from the same point of that line, which is impossible (*Th. XVI, Ex.*); and since it falls at the right or left of C, it will divide AB into two unequal parts, as AD and DB; hence, by the second part of Theorem XIX, $AP' >$ or $< P'B$.

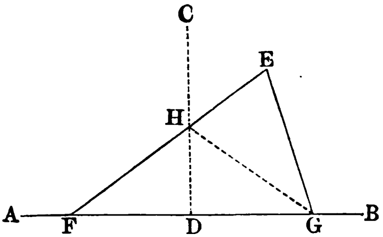
Therefore, *if a perpendicular, etc.*

QUERIES.

1. What previous Theorems are used in the above demonstration?
2. What are the essential points in this demonstration? (1) $AC = BC$;

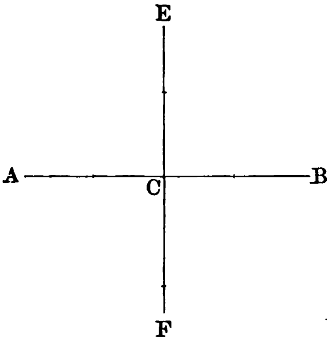
∴ (2) $AP = BP$; (3) $P'D$ cannot pass through C; (4) $AD > DB$; ∴ (5) $AP' > P'B$.

EXERCISES.



1. Prove that, if from a point without the perpendicular, two oblique lines be drawn to the given line at equal distances from the foot of the perpendicular, the oblique line which cuts the perpendicular is the longer; that is $EF > EG$. (*See Th. III.*)

2. Prove that, if a straight line has two of its points equally distant from the extremities of another line, it will be perpendicular to that line at its middle point.

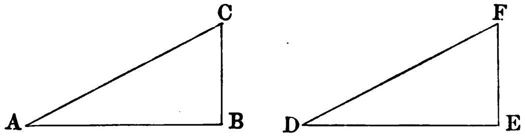


Suggestion.—By the Theorem just proved, E F, a perpendicular through the middle point of A B, includes *all* the points equally distant from A and B; hence, it includes the *two* points of the *supposed* perpendicular; hence, *etc.* (*See Th. XIV, Ex.*)

3. In the plane of a line, what is the locus of points equally distant from the extremities of that line?

THEOREM XXI.

If two right triangles have the hypotenuse and a side of the one equal to the hypotenuse and a side of the other, they will be equal in all their parts.



Let the triangles A B C and D E F have the hypotenuse $AC = DF$, and the side $BC = EF$; then will the two triangles be equal in all their parts.

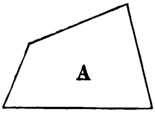
For, let D E F be applied to A B C so that the angle E shall coincide with the angle B (*Ax. 6*), E F coinciding with its equal B C, and E D taking the *direction* of B A. Now, D must fall to the right of A, or to the left of A, or upon A. If it falls to the right, $DF < AC$; if it falls to the left, $DF > AC$ (*Th. XIX*); both of which conclusions are contrary to the supposition,— $AC = DF$: hence, D falls upon A; D F coincides with A C (*Ax. 7*); and the two triangles are coincident.

Therefore, *etc.*

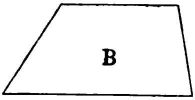
DEFINITIONS.

1. A polygon of four sides is called a **Quadrilateral.** Quadrilaterals are divided into three classes: *trapeziums*, *trapezoids*, and *parallelograms.*

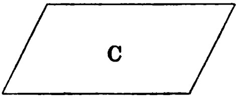
2. A **Trapezium** is a quadrilateral having no two sides parallel, as A.



3. A **Trapezoid** is a quadrilateral having two of its opposite sides parallel, as B.

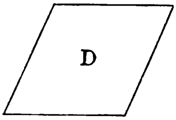


4. A **Parallelogram** is a quadrilateral having its opposite sides parallel, as C. Parallelograms are divided into *rhomboids* and *rectangles.*

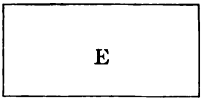


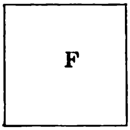
5. A **Rhomboid** is a parallelogram whose angles are all oblique, as C above.

6. A **Rhombus** is an equilateral rhomboid, as D.



7. A **Rectangle** is a parallelogram whose angles are all right angles, as E.



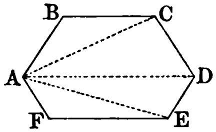


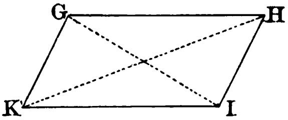
8. A **Square** is an equilateral rectangle, as F.

9. A polygon of five sides is called a **Pentagon**; of six sides, a **Hexagon**, *etc.*

10. An **Equiangular Polygon** is a polygon whose angles are all equal.

11. The **Diagonal** of a polygon is a line joining the vertices of two angles not consecutive as A C, *etc.*



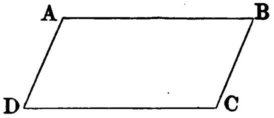


QUERIES.

1. Is a square a parallelogram?
2. Is a parallelogram a square?
3. Is a rectangle a parallelogram?
4. Is a parallelogram a rectangle?
5. As between rectangle and square, which is the species and which the genus?
6. Into how many triangles is a quadrilateral divided by a diagonal?
7. The sum of the angles of a quadrilateral, therefore, is equal to what?
8. If the angles of a quadrilateral are all equal, what is the value of each?

THEOREM XXII.

The opposite angles of a parallelogram are equal.

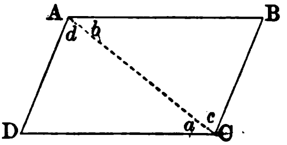


For, the sides of the angles A and C are, by definition, parallel; hence A = C (*Th. XI*). For the same reason, D = B.

Therefore, *etc.*

THEOREM XXIII.

The opposite sides of a parallelogram are equal.



For, draw the diagonal A C; then, in the two triangles A D C and A B C, we have the side A C common; the angle a equal to the angle b, and the angle d equal to the angle c; hence, the triangles are equal in all their parts; hence, A B = D C, and A D = B C.

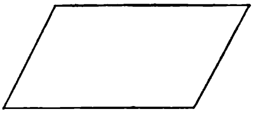
Therefore, *etc.*

Cor.—*Two parallel lines included between two other parallels are equal.*

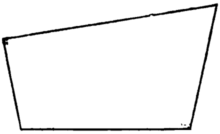
QUERIES.

1. $a = b$, why?
2. $c = d$, why?
3. How may the above Theorem be stated hypothetically?
4. What is the Conclusion?

EXERCISES.



1. Prove, (*Th. X*, *Cor. I*) that the opposite angles of a parallelogram are equal.

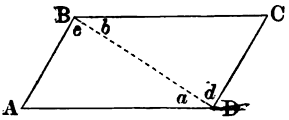


2. Prove (*Th. XII*) that the sum of the angles of a quadrilateral equals four right angles.

3. If one angle of a parallelogram is a right angle, all the others are right angles.

THEOREM XXIV.

If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.



For, draw the diagonal B D; then the triangles A B D and B D C have all the sides of the one equal to the corresponding sides of the other, each to each; hence they are equal in all their parts (*Th. XVII*); the angle *a* equals its alternate angle *b*; and, consequently, A D and B C are parallel (*Th. IX*, *Ex.*). For a similar reason, *c* being equal to *d*, A B and D C are parallel. Therefore, by definition, A B C D is a parallelogram.

Therefore, *if*, *etc.*

QUERIES.

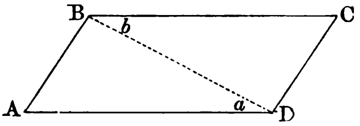
1. What is the *particular statement* of the above Theorem; that is, the statement with reference to the figure ?

2. In the two triangles of this Theorem, which, by hypothesis, are the equal sides?

3. A = C, why ? (*Th. XXII.*)

THEOREM XXV.

If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.



Let A B C D be a quadrilateral having B C equal and parallel to A D; then will the figure be a parallelogram.

For, draw the diagonal B D. Then,

B C = A D - - - - - - *Hypothesis.*

B D is common - - - - - *Construction.*

Included angle *a* = included angle *b* - - *Th. X*, *Ex. 1.*

∴ triangle A B D = triangle B D C - - *Th. IV.*

∴ A B = D C.

Hence the figure is a parallelogram - - - - *Th. XXIV.*

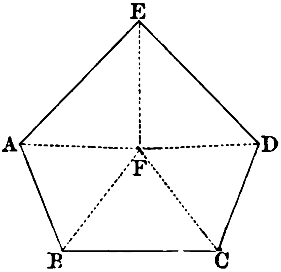
Therefore, *if*, *etc.*

QUERY.

How does the figure meet the conditions of Theorem XXIV?—*A D = B C by hypothesis; A B = D C, as just proved.*

THEOREM XXVI.

The sum of the interior angles of a polygon, plus four right angles, is equal to twice as many right angles as the polygon has sides.



For, take any polygon, as A B C D E. If from any point within it, as F, lines be drawn to the vertices of the polygon, there will be a triangle for every side; for five sides, five triangles; for six sides, six triangles; for seven sides, seven triangles - - - - - - - - ; for n sides, n triangles; that is, the number of triangles = the number of sides.

Now, the value of the angles of one triangle is 2 R (*Th. XII*); of two triangles, *twice* 2 R; of three triangles, *three* times 2 R - - - ; of n triangles, n times 2 R, or 2 Rn; that is, the sum of the angles of all the triangles = 2 Rn. Denote this sum by S; then

$$(1)\quad S = 2\,\mathrm{R}\,n.$$

But S includes the angles at F and the angles at the bases of the triangles; but the angles at F are together equal to 4 R (*Th. II, Cor. II*), and the angles at the bases are together equal to the sum of the interior angles of the polygon (*Ax. 5*). Denote this latter sum by S'; then

$$(2)\quad S' + 4\,\mathrm{R} = S;$$

$$\therefore\ (3)\quad S' + 4\,\mathrm{R} = 2\,\mathrm{R}n.$$

Therefore, *the sum of, etc.*

Cor.—From (3) we have

$$(4)\quad S' = 2\,\mathrm{R}\,n - 4\,\mathrm{R}.$$

Factoring the second member,

$$(5)\quad S' = 2\,\mathrm{R}\,(n-2);$$

that is, ***the sum of the interior angles of a polygon is equal to two right angles taken as many times as the polygon has sides less two.***

CHAPTER III.

RATIO AND PROPORTION.

DEFINITIONS.

1. **Ratio** is the relation between two quantities of the same kind, and is expressed by the quotient of the first divided by the second.

Thus, the ratio of a to b is $\frac{a}{b}$.

2. The two quantities compared are called the **Terms** of the ratio; the first, the **Antecedent**, and the second, the **Consequent.** The antecedent and consequent together constitute a **Couplet.**

3. The sign of a ratio is the colon. A ratio is indicated by placing the sign between its terms, or by writing the antecedent over the consequent.

Thus, $a : b$, or $\frac{a}{b}$.

4. A **Proportion** is an equality of ratios. The equality is indicated by the double colon.

Thus, $a : b :: c : d$ is a proportion, and is read, a is to b as c is to d. It is equivalent to

$$\frac{a}{b}=\frac{c}{d}.$$

The terms of a proportion are four. The first and second terms form the first couplet; the third and fourth, the second couplet.

5. The first and fourth terms are called the **Extremes**; the second and third, the **Means.**

The first and third terms are antecedents; the second and fourth, consequents.

6. Of four proportional quantities, the last is called a **Fourth Proportional** to the other three, taken in order.

Thus, if $a : b :: c : d$, d is a fourth proportional to a, b, and c.

7. *Three* quantities are proportional when the ratio of the first to the second equals the ratio of the second to the third.

Thus, $a : b :: b : c$.

The middle term, b, is here called a **Mean Proportional** between the other two; and c, a **Third Proportional** to a and b.

8. A **Continued Proportion** is one in which several ratios are successively equal to each other; as

$$a : b :: c : d :: e : f :: g : h, \text{ etc.}$$

9. Four quantities are in proportion by **Alternation** when antecedent is compared with antecedent and consequent with consequent.

Thus, if $a : b :: c : d$, by alternation we have $a : c :: b : d$.

10. Four quantities are in proportion by **Inversion** when the consequents are taken for antecedents, and the antecedents for consequents.

Thus, if $a : b :: c : d$, by inversion we have $b : a :: d : c$.

11. Four quantities are in proportion by **Composition** when the sum of the terms of each couplet is compared with either antecedent or consequent.

Thus, if $a : b :: c : d$, by composition we have

$$a + b : a :: c + d : c, \text{ or}$$
$$a + b : b :: c + d : d.$$

12. Four quantities are in proportion by **Division** when the difference of the terms of each couplet is compared with either antecedent or consequent.

Thus, if $a : b :: c : d$, by division we have

$$a - b : a :: c - d : c, \text{ or}$$
$$a - b : b :: c - d : d.$$

13. Four quantities are in proportion by **Composition and Division** when the sum of the terms of each couplet is compared with their difference.

Thus, if $a : b :: c : d$, by composition and division we have

$$a + b : a - b :: c + d : c - d.$$

14. Equimultiples of several quantities are the products of those quantities by the same multiplier.

Thus $n\,a$ and $n\,b$ are equimultiples of a and b.

Note.—The object of this chapter is to communicate the principles of proportion, if they are not known; to refresh the memory, if they are; and to furnish a convenient reference when needed. The student should fix clearly in his mind the idea that a proportion is an *equation*—that the expression $a : b :: c : d$ means simply $\frac{a}{b} = \frac{c}{d}$, and conversely.

THEOREM I.

If four quantities are in proportion, the product of the extremes equals the product of the means.

Let $a : b :: c : d$; then will $a \times d = b \times c$.

For (1) $\frac{a}{b} = \frac{c}{d}$ by definition; and clearing of fractions,

(2) $a\,d = b\,c$.

Therefore, *etc.*

THEOREM II.

If the product of two quantities equals the product of two other quantities, two of them may be made the means, and the other two the extremes of a proportion.

Let (1) $a \times d = b \times c$; then will

$$a : b :: c : d.$$

For, (1) $\div\ b\,d =$ (2) $\frac{a}{b} = \frac{c}{d}$; $\therefore$ by definition,

$$a : b :: c : d.$$

Therefore, *etc.*

EXERCISES.

1. A mean proportional between two quantities equals the square root of their product.

2. What proportion can be formed from the equation, $c \times d = a^2 - b^2$?

THEOREM III.

If four quantities are in proportion, they will be in proportion by alternation.

Let $a : b :: c : d$; then will $a : c :: b : d$.

For, (1) $a \times d = b \times c$ - - - - - - - - *Th. I.*

(1) $\div\ d c =$ (2) $\frac{a}{c} = \frac{b}{d}$; whence, $a : c :: b : d$.

Therefore, *etc.*

THEOREM IV.

If four quantities are in proportion, they will be in proportion by inversion.

Let $a : b :: c : d$; then will $b : a :: d : c$.

For, (1) $a \times d = b \times c$; and taking the factors of the second product for the extremes (*Th. II*), we have

$$b : a :: d : c.$$

Therefore, *etc.*

THEOREM V.

If four quantities are in proportion they will be in proportion by composition.

Let $a : b :: c : d$; then will $a + b : a :: c + d : c$.

For, (1) $a \times d = b \times c$, and (2) $a \times c = a \times c$.

(1) + (2) = (3) $a d + a c = b c + a c$, whence,

(4) $a (c + d) = c (a + b)$.

$\therefore$ $a + b : a :: c + d : c$.

Therefore, *etc.*

EXERCISES.

1. If $a : b :: c : d$, prove that $a + b : b :: c + d : d$.

2. Given, $3 : 12 :: 5 : 20$; required the proportion by composition.

3. What proportion can be formed from the equation $a(b + c) = b(a + d)$?

THEOREM VI.

If four quantities are in proportion, they will be in proportion by division.

Let $a : b :: c : d$; then will $a - b : a :: c - d : c$.

For, (1) $ad = bc$, and (2) $ac = ac$.

(2) − (1) = (3) $ac - ad = ac - bc$;

$\therefore$ (4) $a(c - d) = c(a - b)$.

$\therefore$ $a - b : a :: c - d : c$.

Therefore, *etc.*

THEOREM VII.

If two proportions have a ratio of the one equal to a ratio of the other, the remaining terms are proportional.

Let $a : b :: c : d$, and $a : b :: e : f$; then will $c : d :: e : f$.

For, (1) $\frac{a}{b} = \frac{c}{d}$, and (2) $\frac{a}{b} = \frac{e}{f}$; hence

(3) $\frac{c}{d} = \frac{e}{f}$.

$\therefore$ $c : d :: e : f$.

Therefore, *etc.*

THEOREM VIII.

If four quantities are in proportion, they will be in proportion by composition and division.

Let $a : b :: c : d$; then will $a + b : a - b :: c + d : c - d$.

For, (1) $a + b : a :: c + d : c$, whence (*Th. III*),

(2) $a + b : c + d :: a : c$; and (*Th. VI*),

(3) $a - b : a :: c - d : c$, whence

(4) $a - b : c - d :: a : c$. $\therefore$, from (2) and (4),

(5) $a + b : a - b :: c + d : c - d$ - - - *Th. VII.*

Therefore, *etc.*

EXERCISE.

If the antecedents of two proportions are in proportion, the consequents will be proportional.

THEOREM IX.

If one antecedent is greater or less than its consequent, the other antecedent is greater or less than its consequent.

Let $a : b :: c : d$. If $a >$ or $< b$, then $c >$ or $< d$.

For, $\frac{a}{b} = \frac{c}{d}$: and, if $a > b$, $\frac{a}{b} > 1$; hence its equal, $\frac{c}{d}$, is greater than one; hence the numerator is greater than the denominator; that is, $c > d$.

If $a < b, \frac{a}{b} < 1$; hence $\frac{c}{d} < 1$: hence $c < d$.

Therefore, *etc.*

EXERCISES.

1. If the antecedents of a proportion are equal, the consequents are equal.
2. If in the proportion, $a : b :: c : d$, $a >$ or $< c$, prove that $b >$ or $< d$.
3. State the above as a principle.

THEOREM X.

Two quantities have the same ratio as their equimultiples.

Let a and b be any two quantities; then $\frac{a}{b}$ will denote their ratio. If we multiply both terms of this fraction by any quantity, as m, the resulting fraction must have the same value; that is,

$$\frac{a}{b} = \frac{ma}{mb}; \text{ whence}$$

$$a : b :: ma : mb.$$

Therefore, *etc.*

EXERCISES.

1. If $a : b :: c : d$, show that $ma : mb :: nc : nd$.
2. State the above as a general truth.

THEOREM XI.

If two quantities be increased by quantities having the same ratio, the resulting quantities will have the same ratio.

Let $a : b :: c : d$, in which c and d have the same ratio as a and b; then will

$$a : b :: a + c : b + d.$$

For, (1) $a : c :: b : d$ - - - - - - - - *Th. III.*

(2) $a + c : a :: b + d : b$ - - - - - - - *Th. V.*

$\therefore$ (3) $a + c : b + d :: a : b$, or $a : b :: a + c : b + d$.

Therefore, *etc.*

EXERCISES.

1. If two quantities be diminished by quantities having the same ratio, the results will have the same ratio.
2. If $a : b :: ma : mb$, show that
   $a : b :: a + ma : b + mb$, in which m is integral or fractional.
3. If $a : b :: c : d$, show that
   $a + ma : b + mb :: c + nc : d + nd$.
4. State '2' and '3' in general terms.

THEOREM XII.

In a continued proportion, the sum of the antecedents is to the sum of the consequents as any antecedent is to its corresponding consequent.

Let $a : b :: c : d :: e : f :: g : h$;
then will $a + c + e + g + etc. : b + d + f + h + etc. :: a : b$.

For, (1) $ad = bc$,

(2) $af = be$,

(3) $ah = bg$,

and (4) $ab = ab$; *etc.* - - - - - - - - - - - - *Th. I.*

Adding and factoring,

(5) $a(b + d + f + h + etc.) = b(a + c + e + g + etc.)$;

$\therefore$ $a + c + e + g + etc. : b + d + f + h + etc. :: a : b$.

Therefore, *etc.*

THEOREM XIII.

The products of the corresponding terms of two proportions are in proportion.

Let $a : b :: c : d$, and $e : f :: g : h$; then will $ae : bf :: cg : dh$.

For, (1) $\frac{a}{b} = \frac{c}{d}$, and (2) $\frac{e}{f} = \frac{g}{h}$.

(1) × (2) = (3) $\frac{ae}{bf} = \frac{cg}{dh}$. $\therefore ae : bf :: cg : dh$.

Therefore, *etc.*

THEOREM XIV.

Like powers of the terms of a proportion are in proportion.

Let $a : b :: c : d$; then will $a^n : b^n :: c^n : d^n$.

For, (1) $\frac{a}{b} = \frac{c}{d}$.

$(1)^n =$ (2) $\frac{a^n}{b^n} = \frac{c^n}{d^n}$ - - - - - - - - - - - - - *Ax. 2.*

$\therefore a^n : b^n :: c^n : d^n$, in which n denotes any number, integral or fractional.

Therefore, *etc.*

EXERCISES.

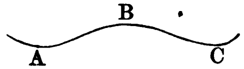
1. Like roots of proportional quantities are proportional.
2. The quotients of the corresponding terms of two proportions are proportional.
3. Find a mean proportional to m and n.
4. If the ratio of $3a$ to $2b$ is $\frac{3}{4}$, what is the ratio of a to b?
5. Given, $a^2 - b^2 : a + b :: (a-b)^2 : x$; required x.
6. If $a : b :: b : c :: c : d$, show that $a : d :: a^3 : b^3$.

CHAPTER IV.

CIRCLES.

DEFINITIONS.

1. A **Curved Line**, or simply a **Curve**, is a line which changes its direction at every point, as A B C.



2. A **Circle** is a portion of a plane bounded by a curve, every point of which is equally distant from a point within.

3. The curve which bounds the circle is called its **Circumference.**

4. The point within, which is equally distant from every point of the circumference, is called the **Centre.**

5. Any straight line drawn from the centre to the circumference is called a **Radius.**

6. A **Diameter** of a circle is a straight line passing through the centre, and terminated both ways by the circumference.

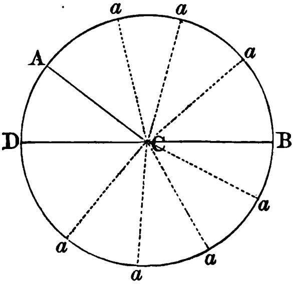


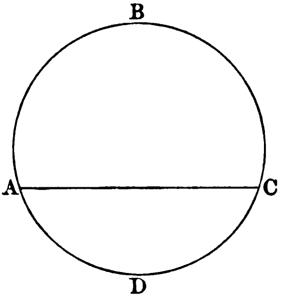
Illustration.—A circle may be conceived as the path of a line, A C, one end of which, C, remains fixed at the same point, while the other end, as the line revolves, moves about this point in the plane, say, of the paper. The curved line, A *a* B *a* D, is the *circumference*, the path of the point A; the space within the circumference is the *circle;* C is the *centre*, A C is the *radius*, and D B the *diameter.*

7. A **Chord** is any straight line drawn from one point of the circumference to another.

8. Any portion of the circumference is an **Arc.**

Every chord subtends two arcs whose sum is the circumference.

9. That portion of the circle included between the arc and its chord is called a **Segment.**



Thus, A C is a *chord;* A B C and A D C, *arcs;* and A C D and A C B, *segments.*

It is seen that every chord divides the circumference into two arcs, and the circle into two segments, which are, in general, unequal. In any given case, the less is understood, unless the contrary is stated.

10. An arc equal to one half the circumference, is called a **Semicircumference.**

11. A segment equal to one half the circle, is called a **Semicircle.**

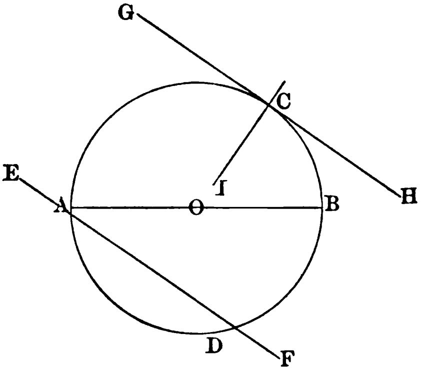
12. A straight line which *cuts* the circumference in two points, is called a **Secant.**

13. A straight line which touches the circumference, but cannot cut it, when produced, is called a **Tangent.**

14. The point at which the straight line touches the circumference, is called the **Point of Contact,** or **Point of Tangency.**

15. A straight line drawn through the point of contact, perpendicular to the tangent, is called the **Normal.**

Thus, in the annexed diagram, A D B is a *semicircumference;* A B D, a *semicircle;* E F, a *secant;* G H, a *tangent;* C, the *point of contact;* and C I, the *normal.*



A circle may be read by the single letter at its centre, as the circle O.

16. The **Converse** of a theorem is formed by transposing its Hypothesis and its Conclusion. Thus the converse of,

If A is B, then C will be D,

is, *If C is D, then A will be B;* in which the hypothesis of the first is the conclusion of the second, and the conclusion of the first, the hypothesis of the second.

Similarly, the converse of, *The angles at the base of an isosceles triangle are equal* (meaning, *if a triangle has two sides equal,* the angles, etc.), is, *If a triangle has two of its angles equal, the sides opposite them are equal.*

17. **Q. E. D.** stands for *quod erat demonstrandum* (*which was to be proved*), and is written at the end of a theorem to denote that the truth of the theorem has been demonstrated.

Cor. I.—*All radii of the same circle are equal.*

Cor. II.—*All diameters of the same circle are equal, and each is double the radius.*

Cor. III.—*Circles having equal radii are equal; and, conversely, equal circles have equal radii.*

QUERIES.

1. What is the difference between a diameter and a chord?
2. Wherein does a semicircumference differ from an arc?
3. A semicircle from a segment?
4. A secant from a tangent?

EXERCISES.

1. Is a circle described by a point or by a line?

2. Two roads run by a piece of ground in the form of a circle, the diameter of which is 100 rods; one road runs within 40 rods of the centre, and the other within 50 rods. How do the roads lie with reference to the circle?

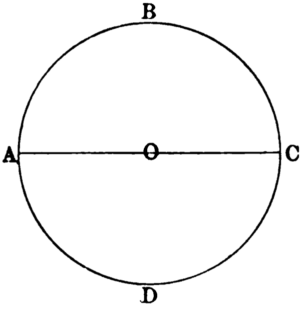
3. What is the locus of all the points in a plane at a given distance from a fixed point of that plane?

4. What is the converse of *Th. VII, Ch. II? of Th. X?*

5. What Theorem of *Ch. II* is the converse of *Th. I?*

THEOREM I.

Any diameter bisects the circle and its circumference.



Let A B C D be a circle, and A C any diameter; then will A C bisect the circle and also its circumference.

For suppose A C D to revolve about A C as an axis, till it falls upon the plane of A C B; then, since every point in A D C is at the same distance from the centre O as every point in A B C, the two arcs will coincide; otherwise there would be points in one or the other unequally distant from the centre. Hence, too, the segments will coincide.

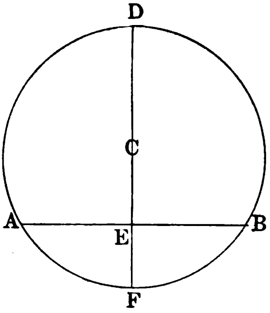
$\therefore$ A C D = A C B, and A D C = A B C. *Q. E. D.*

Scholium.—A line which divides a figure in this manner, is said to divide it *symmetrically*, and the figure is said to be *symmetrical* with respect to such line.

Note.—Let the student notice carefully the distinction between the reading of an arc and that of a segment. In the former, the first and last letters are at the extremities of the chord; in the latter, either letter of the chord is read *between* the other two. An arc may also be read by two letters with a curved mark over them, or under them, as the case may require.

THEOREM II.

A diameter perpendicular to a chord bisects the chord and also the subtended arc.



Let C be a circle, and D F a diameter perpendicular to the chord A B; then will A E = B E, and $\underset{\smile}{AF} = \underset{\smile}{BF}$.

For, revolve A F D about D F as an axis till it coincides with B F D (*Th. I*); then, since A B is perpendicular to D F, E A will take the *direction* of E B, and therefore A will fall upon B; for, if it fell to the right or left of B, the two semi-circumferences would not coincide; and since the semicircumferences coincide, $\underset{\smile}{\mathrm{AF}}$ must coincide with $\underset{\smile}{\mathrm{BF}}$. ∴ A E = B E, and $\underset{\smile}{\mathrm{AF}} = \underset{\smile}{\mathrm{BF}}$. *Q. E. D.*

Cor. I.—*A straight line perpendicular to a chord at its middle point passes through the centre of the circle.*

For, it contains *all* the points that are equally distant from A and B (*Ch. II, Th. XX*); hence it contains C, and therefore passes through the centre of the circle.

Cor. II.—*A line that satisfies two of the five conditions,—passing through centre of circle, middle of chord, middle points of arcs subtended, being perpendicular to chord,—satisfies the others.*

For, by the Theorem, the line would have at least two points in common with the diameter perpendicular to the chord, and hence would coincide with it (*Ch. II, Th. XIV, Ex. 1*).

EXERCISES.

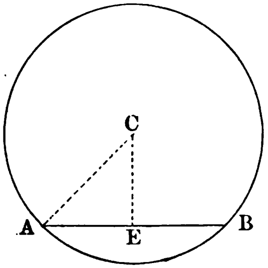
1. The radius perpendicular to a chord bisects the angle formed by the lines drawn from any point of the radius to the extremities of that chord.

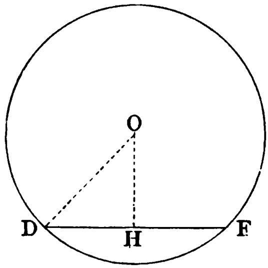
2. Where is the point of intersection of the perpendiculars to two chords at their middle points?

3. What is the locus of the middle points of a system of parallel chords?

THEOREM III.

In the same circle or in equal circles, equal chords are equally distant from the centre.





Let C and O be two equal circles, A B and D F two equal chords; then will A B and D F be equally distant from the centre.

For, let fall the perpendiculars C E and O H, and draw the radii C A and O D.

In the right triangles A C E and D O H, A E = D H (*Th. II*); and A C = D O, being radii of equal circles.

∴ C E = O H - - - - - - - - - - - - *Ch. II, Th. XXI.*

Therefore A B and D F are equally distant from the centre.

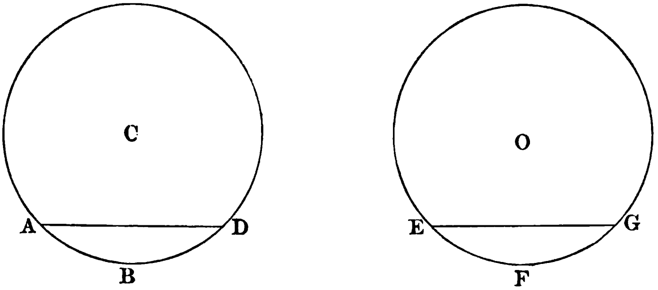
Q. E. D.

QUERY.

Why are A C and D O oblique lines?

THEOREM IV.

In the same circle or in equal circles, equal arcs are subtended by equal chords.



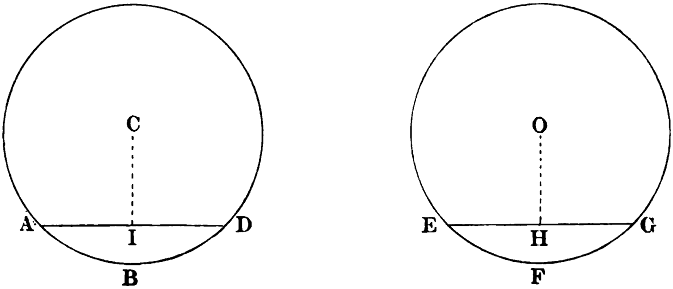
Let C and O be two equal circles, A B D and E F G two equal arcs; then will A D = E G.

For, let C be applied to O so that the equal arcs shall coincide; then A will fall upon E, and D upon G; ∴ A D = E G - *Ax. 7.*

Q. E. D.

THEOREM V.

In the same circle or in equal circles, if the chords are equal the arcs are equal.

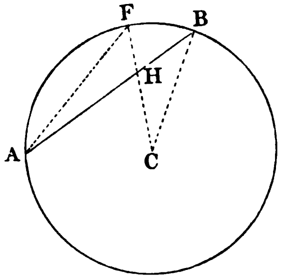


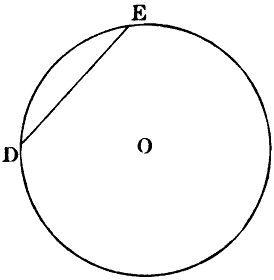
Let C and O be two equal circles, A D and E G two equal chords; then will A B D = E F G.

For, let fall the perpendiculars, C I, O H ; and let the circle C be applied to the circle O so that C shall fall upon O and C I take the *direction* of O H. Since C I and O H are equal (*Th. III*), they will coincide; and A D, which is perpendicular to C I, will then be perpendicular to O H; if so, it will coincide with E G, otherwise there would be two perpendiculars to a line from a point in that line; and since A falls upon E, and D upon G, *all* the points of A B D must fall upon the corresponding points of E F G, otherwise there would be points unequally distant from the common centre, which is impossible, since the circles are equal. ∴ A B D = E F G. *Q. E. D.*

THEOREM VI.

In the same circle or in equal circles, the less of two arcs has the shorter chord.





Let C and O be two equal circles, and $\overset{\frown}{DE} < \overset{\frown}{AB}$; then will $DE < AB$.

For, cut off $\overset{\frown}{AF} = \overset{\frown}{DE}$, and draw AF, CF, and CB.

(1) $BC < CH + HB$; but

$BC = CF$; $\therefore$ (2) $CF < CH + HB$.

(2) $- CH =$ (3) $HF < HB$.

(3) $+ AH =$ (4) $AH + HF < AB$; but

$AF < AH + HF$; $\therefore$ (5) $AF < AB$; but

$AF = DE$ (*Th. IV*); $\therefore$ $DE < AB$. *Q. E. D.*

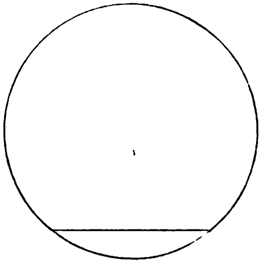
QUERIES.

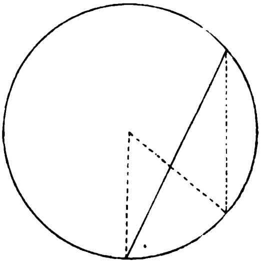
1. What theorem is involved in (1)?
2. What algebraic principles are involved in (3) and (4)?
3. What axiom in (4)?

EXERCISES.

1. State the converse, in hypothetical form, of *Th. VI.*

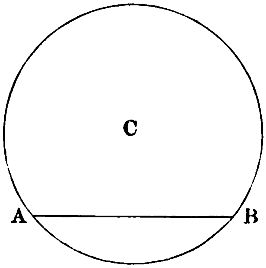
2. Prove *Th. VI* from the annexed diagram, without any notation,—letters or numbers.

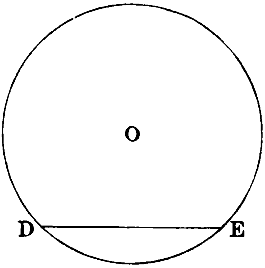




THEOREM VII.

In the same circle or in equal circles, the less chord subtends the less arc.





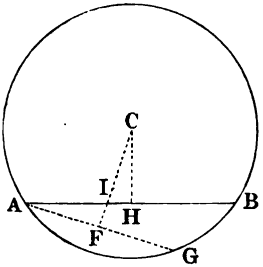
Let C and O be two equal circles, and $DE < AB$; then will $\underset{\smile}{DE} < \underset{\smile}{AB}$.

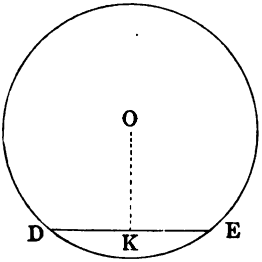
For, if not, $\underset{\smile}{DE}$ must either $= \underset{\smile}{AB}$ or be $> \underset{\smile}{AB}$. If $\underset{\smile}{DE} = \underset{\smile}{AB}$, $DE = AB$ (*Th. IV*); if $\underset{\smile}{DE} > \underset{\smile}{AB}$, $DE > AB$ (*Th. V*). Both of these conclusions are contrary to the supposition. Hence, both are false; and therefore, $\underset{\smile}{DE}$ can neither be equal to, nor greater than, $\underset{\smile}{AB}$. $\therefore \underset{\smile}{DE} < \underset{\smile}{AB}$. *Q. E. D.*

THEOREM VIII.

In the same circle or in equal circles, the less of two chords is at the greater distance from the centre.

Let C and O be two equal circles, having the chord $DE <$ AB; then will $OK > CH$.





For, since $DE < AB$, $\underset{\smile}{DE} < \underset{\smile}{AB}$ - - - - - - *Th. VII.*

Cut off $\underset{\smile}{AG} = \underset{\smile}{DE}$, and draw AG. Let CH be perpendicular to AB, CF to AG, and OK to DE.

$CI > CH$ (*Ch. II, Th. XIX*);

but $CF > CI$; $\therefore CF > CH$; but $CF = OK$ (*Th. III*);

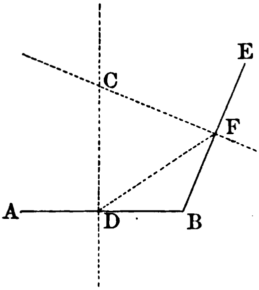
$\therefore OK > CH$. *Q. E. D.*

THEOREM IX.

Through any three points not in the same straight line one circumference can be made to pass, and but one.

Let A, B, and E be any three points not in the same straight line, then can one circumference be made to pass through them, and but one.

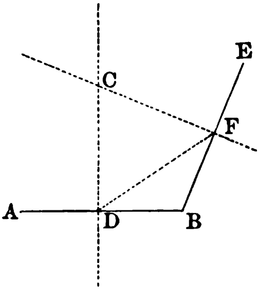
For, draw AB and BE, and at their middle points, D and F, erect perpendiculars; join D and F. By construction,



$CDB + CFB = 2R$, $\therefore$

$CDF + CFD < 2R$ - - - - - - - - - - - - *Ax. 5.*

Hence C D and C F will meet if sufficiently produced (*Ch. II, Th. XXVII Ex.* 7.)



Let C be their point of intersection.

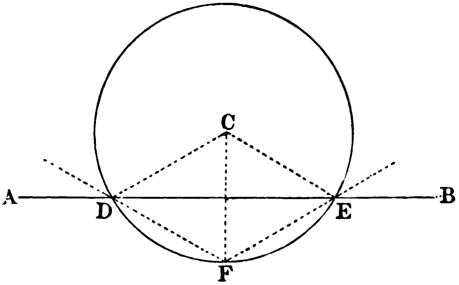
Now, the distance from C to B = distance from C to A; but distance from C to B = distance from C to E (*Ch. II, Th. XX~~IV~~*); hence distance from C to A = distance from C to E, and C is equally distant from A, B, and E (*Ax. 1*); and, if from C as a centre, with a radius C A, a circle be described, its circumference will pass through A, B, and E. Moreover, as this circumference is the locus of *all* the points in the plane at the given distance C A from C, any other circumference through A, B, and E, must coincide with it. *Q. E. D.*

Cor.—*Two circumferences can intersect in only two points.*

For, if they intersect in a third, they will coincide.

THEOREM X.

A straight line can intersect a circumference in only two points.



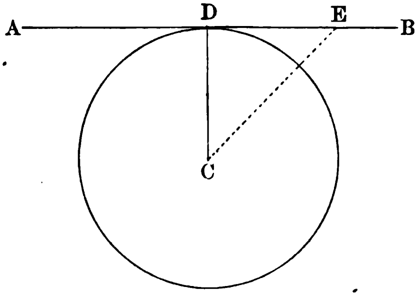
Let C be a circle, and A B a straight line; then can A B cut the circumference in only two points, as D and E.

For, suppose it to intersect in a third, as F. Draw the radii C D, C F, and C E; then, since these radii are equal, and since the line is supposed to continue straight, we would have three equal lines drawn from the same point to the same straight line, which is impossible.

Therefore, it is false that A B intersects in a third point; therefore, it intersects in only two. *Q. E. D.*

THEOREM XI.

A straight line perpendicular to a radius at its extremity, is tangent to the circumference at that point.



Let C be a circle, and A B a straight line perpendicular to the radius C D at its extremity; then will A B have only this point in common with the circumference.

For, suppose it to touch in another point, as E. Draw C E. Since, by supposition, E is on the circumference, C E = C D; ∴ C D E = C E D (*Ch. II, Th. VI*), which is impossible (*Ch. II, Th. XV*).

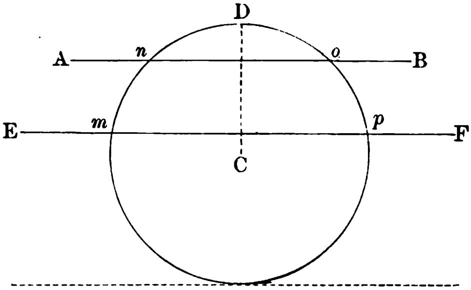
Therefore, A B has no other point than D in common with the circumference, and is therefore tangent at D. *Q. E. D.*

Cor.—*Conversely, a straight line tangent to the circumference at any point, is perpendicular to the radius drawn to that point.*

For, any line, as C E, drawn from the centre to any other point of A B than the point of contact, is longer than C D, a radius; therefore C D is the shortest distance from C to A B; therefore C D is perpendicular to A B; and therefore A B is perpendicular to C D.

THEOREM XII.

Two parallels intercept equal arcs of a circumference.



Let C be a circle, A B and E F two parallel lines; then will $\overset{\frown}{mn} = \overset{\frown}{op}$.

For, draw the radius C D perpendicular to mp; it will also be perpendicular to no, and bisect the subtended arcs, $\overset{\frown}{mp}$ and $\overset{\frown}{no}$ - - - - *Th. II.*

$\therefore \quad \overset{\frown}{m\mathrm{D}} = \overset{\frown}{\mathrm{D}p}$, and $\overset{\frown}{n\mathrm{D}} = \overset{\frown}{\mathrm{D}o}$.

$\therefore \quad \overset{\frown}{m\mathrm{D}} - \overset{\frown}{n\mathrm{D}} = \overset{\frown}{\mathrm{D}p} - \overset{\frown}{\mathrm{D}o}$.

Hence $\overset{\frown}{mn} = \overset{\frown}{op}$. *Q. E. D.*

Scholium 1.—This proposition is evidently true whatever be the position of the parallel lines,—whether they are both secants, or one or both are tangents. Suppose E F to remain fixed, and let A B move toward D, but in such a manner as to continue parallel to E F. It is plain that in every position of A B, $\overset{\frown}{mn} = \overset{\frown}{op}$, and this will be true when n and o coincide with D, or when A B becomes tangent. If we suppose E F, in turn, to move downward till it reaches the point of tangency, it is equally plain that the two arcs continue equal.

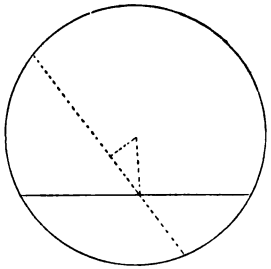
Scholium 2.—The tangent may be regarded as a secant whose two points of intersection are coincident.

Note.—The student should realize, at every step, that definitions, axioms, theorems, and corollaries, are the *tools* with which he is to work,—the materials, the instruments, upon which he must depend in the performance of a given task. Without them he is powerless. In proportion as they are held distinctly and securely in memory—are *available*, will be his satisfaction and success in the solution of problems and the demonstration of theorems.

EXERCISES.

1. In the same circle or in equal circles, chords equally distant from the centre are equal; and, of two chords unequally distant from the centre, that is the greater, whose distance from the centre is the less. (*Ths. III* and *VIII.*)

2. The least chord that can be drawn in a circle through a given point is the chord perpendicular to the line joining the given point and the centre.



3. There can be but one tangent at a given point of the circumference.

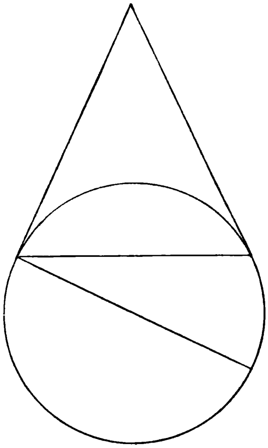
4. The normal at every point passes through the centre of the circle.

5. Any two tangents to a circle, drawn from a common point, are equal.

6. If from a point without the circumference, tangents be drawn to the extremities of a chord, the angle contained by the tangents is twice the angle contained by the chord and the diameter passing through either extremity of the chord.

7. A diameter is greater than any other chord.

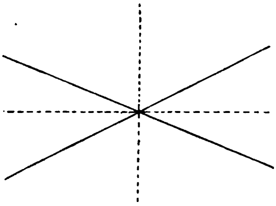
8. The lines which bisect at right angles the sides of a triangle all meet in one point. (*Th. IX.*)



9. Find the locus of the centres of all circles that pass through two fixed points.

10. Find the locus of a point at a given distance from a given circle.

11. Find the locus of a point equidistant from two intersecting straight lines.



DEFINITIONS.

1. Two circles are **concentric** when they have the same centre.

2. Two circles are **coincident** when they are concentric and equal.

3. Two circles are **tangent externally** when each is outside the other, with their circumferences in contact.

4. Two circles are **tangent internally** when one is within the other, with their circumferences in contact.

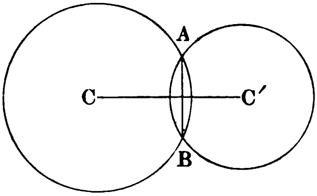
5. Two circles are **external** to each other when each is outside the other, with no point common.

6. One circle is **wholly within** another when it is enclosed by that other, with no point common to their circumferences.

7. Two circles **intersect** when they cut each other.

THEOREM XIII.

If two circles intersect each other, the straight line joining their centres bisects their common chord at right angles.

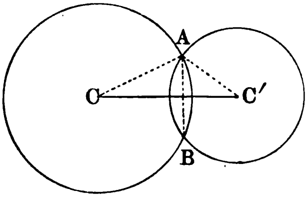


Let C and C′ intersect in the points A and B, C C′ be the line joining their centres, and A B their common chord; then will C C′ bisect A B at right angles.

For, the perpendicular drawn to A B at its middle point will pass through the centres, C and C′, and therefore be coincident with the line joining these centres; hence C C′ bisects A B at right angles. *Q. E. D.*

THEOREM XIV.

If two circles intersect, the distance between their centres is less than the sum, and greater than the difference of their radii.



Let the circles, C and C′, intersect in the points A and B; then will the distance between the centres be less than the sum, and greater than the difference of the radii.

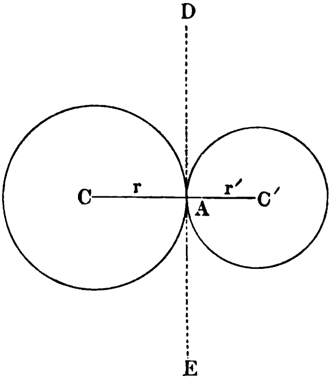
For, neither point of intersection is on C C′, since C C′ bisects the chord joining them, and therefore passes *between* them; hence, if either point of intersection be joined with the centres, the resulting figure is a triangle.

$\therefore$ $CC' < CA + AC'$, and $> CA - AC'$ - *Ch. II, Th. III.*

Q. E. D.

THEOREM XV.

If two circles are externally tangent, the straight line joining their centres passes through the point of tangency.



Let C and C′ be externally tangent at the point A; then will the straight line joining their centres pass through A.

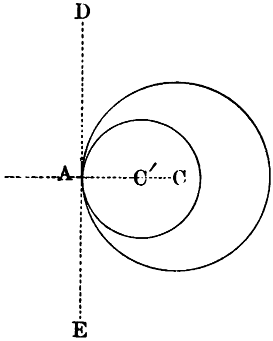
For, through the point of contact draw the common tangent, D E, and to the same point draw the radii, r and r'. Now, r and r' are each perpendicular to D E at A (*Th. IX*), and hence form one and the same straight line (*Ch. II, Th. XIV*), and the only straight line that can pass through the given centres (*Ax. 7*); hence, C, C′, and A, are in the same straight line; hence, the line joining any two, as C and C′, will pass through the third, as A. *Q. E. D.*

Cor. I.—*If two circles are externally tangent, the straight line joining their centres is perpendicular to their common tangent at the point of contact.*

Cor. II.—*If two circles are externally tangent, the distance between their centres is equal to the sum of their radii.*

THEOREM XVI.

If two circles are internally tangent, the straight line joining their centres passes through the point of tangency.



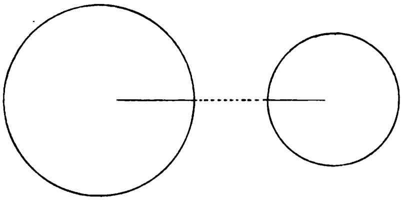
Let C and C′ be two circles tangent at A internally; then will the straight line joining their centres pass through A.

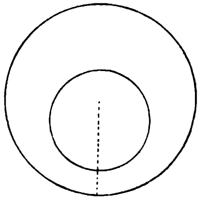
For, the normal at A will pass through the centre of each circle (*Th. XII, Ex. 4*), and therefore be coincident with any straight line joining their centres (*Ax. 7*); hence C, C′, and A, are in the same straight line; and C C′, if produced, will pass through A. *Q. E. D.*

Cor.—*If two circles are internally tangent, the distance between their centres is equal to the difference of their radii.*

It is evident from the diagram, that C C′ = A C − A C′.

Scholium.—If two circles are wholly external to each other, it is plain that the distance between their centres is greater than the sum of their radii; and if one is wholly within the other, this distance is less than the difference of their radii.





EXERCISES.

1. Two circles intersect, if the distance between their centres is less than the sum, and greater than the difference, of their radii.

Suggestion.—For, if they do not intersect, either they are tangent, externally or internally, or they are wholly external, or one is wholly within the other; but if they are tangent, etc. Use the *reductio ad absurdum,* showing that the conditions are inconsistent with any other relation than that of intersection.

2. Two circles are tangent externally, if the distance between their centres is equal to the sum of their radii.

3. Two circles are tangent internally, if the distance between their centres is equal to the difference of their radii.

4. Two circles are wholly external to each other when the distance between their centres is greater than the sum of their radii.

5. One circle is wholly within another when the distance between their centres is less than the difference of their radii.

6. How many circles may be described from the same point?

7. How many circles can be drawn tangent to a given straight line at a given point of that line?

8. What is the locus of their centres?

9. What is the locus of the centres of all the circles having a given radius, and externally tangent to a given circle? (*See Th. XV, Cor. II.*)

10. What is the locus of the centres of all the circles having a given radius, and internally tangent to a given circle?

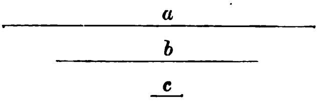
DEFINITIONS.

1. **To measure** a quantity is to find how many times it contains another quantity of the same kind called the **Unit of Measure.**

Thus, to measure a line is to find how many times it contains another line called the *unit of length,* or *linear unit.*

2. The number expressing how many times a quantity contains its unit, is called the **numerical measure** of that quantity.

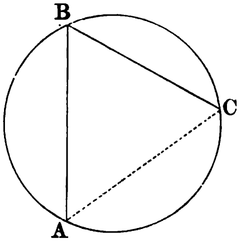
Thus, if c is contained in a 10 times, and in b 6 times, the numerical measure of a is 10, and of b 6; c being 1, the unit of length.



3. A **common measure** of two quantities is a third quantity of the same kind which is contained an exact number of times in each.

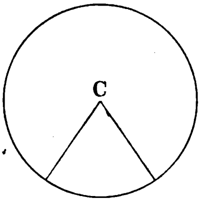
4. Two quantities are **commensurable** when they have a common measure.

5. Two quantities are **incommensurable** when they have no common measure; that is, when no assumed unit is contained in each without a remainder.



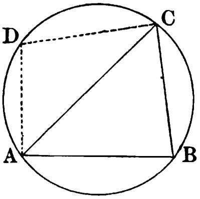
6. An **Inscribed Angle** is an angle whose vertex is in the circumference, and whose sides are chords, as A B C.

7. An angle is **inscribed in a segment** when its vertex is in the arc of the segment, and when its sides pass through the extremities of the subtending chord. Thus, the angle A B C is inscribed in the segment A C B.



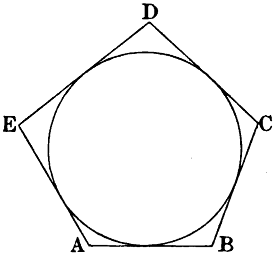
8. A **Central Angle** is one whose vertex is at the centre, and whose sides are radii, as C.

An inscribed or central angle may be said to *stand on* the arc intercepted between its sides.



9. An **Inscribed Polygon** is one whose vertices are all in the circumference; as, the triangle A B C, or the quadrilateral A B C D.

The circle is then said to be *circumscribed* about the polygon.



10. A **Circumscribed Polygon** is one whose sides are all tangent to the circumference; as, A B C D E.

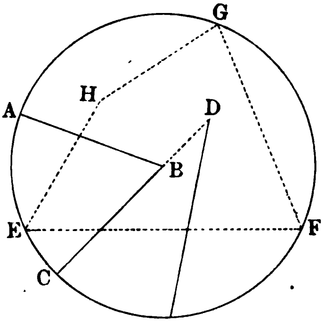
The circle is then said to be inscribed in the polygon.

11. When two angles are together equal to one right angle, each is called the **Complement** of the other.

12. When two angles are together equal to two right angles,

each is called the **Supplement** of the other. (*See Ch. I, p. 5, Defs. 13 and 14.*)

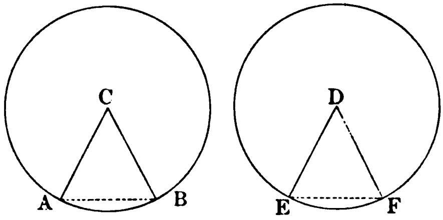
QUERIES.



1. Is A B C an inscribed angle?
2. Is D inscribed or central?
3. Is not E F G H an inscribed polygon?
4. In what segment is the angle E F G inscribed?
5. On what arc does it stand,—E F G or E A G?
6. What is the complement of $\frac{2}{3}$ of a right angle?
7. What is its supplement?

THEOREM XVII.

In equal circles, equal central angles intercept equal arcs of the circumference.

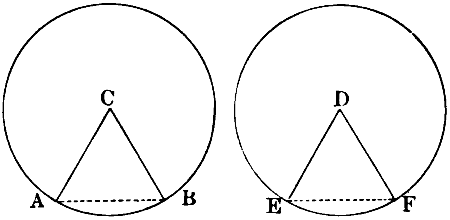


In the equal circles C and D, let A C B = E D F; then will $\underset{\smile}{AB} = \underset{\smile}{EF}$.

For, draw the chords A B and E F: the triangles A C B and E D F are equal in all their parts (why?); hence A B = E F; hence $\underset{\smile}{AB} = \underset{\smile}{EF}$ (*Th. V*). *Q. E. D.*

THEOREM XVIII.

In equal circles, equal arcs are intercepted by equal central angles.



In the equal circles C and D, let $\underset{\smile}{AB} = \underset{\smile}{EF}$; then will A C B = E D F.

For, draw the chords A B and E F: then, as $\underset{\smile}{AB} = \underset{\smile}{EF}$, A B = E F; hence the triangles A C B and E D F are equal; hence the angles C and D are equal. *Q. E. D.*

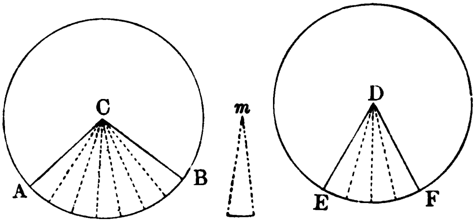
QUERIES.

1. How does it appear that $\underset{\smile}{AB} = \underset{\smile}{EF}$?
2. By what theorem does $AB = EF$?
3. What are the conditions that make the triangle $ACB = EDF$?
4. What is the converse of the above Theorem?
5. The general statement? The particular?
6. What portion of the diagram is construction?

THEOREM XIX.

In equal circles, central angles are proportional to their intercepted arcs.

There may be two cases: the angles may be commensurable, or they may be incommensurable.



First: In the equal circles C and D, let ACB and EDF be commensurable; then will $ACB : EDF :: \underset{\smile}{AB} : \underset{\smile}{EF}$.

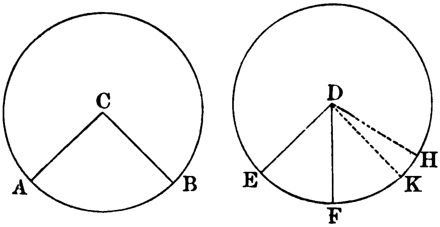
For, suppose *m*, the unit of measure, to be contained in ACB 7 times, and in EDF 4 times; then

(1) $ACB : EDF :: 7 : 4$.

Since the partial angles are equal, being each equal to *m*, they intercept equal arcs (*Th. XVII*); hence $\underset{\smile}{AB}$ is divided into 7 equal parts and $\underset{\smile}{EF}$ into 4; hence,

(2) $\underset{\smile}{AB} : \underset{\smile}{EF} :: 7 : 4$.

$\therefore$ (3) $ACB : EDF :: \underset{\smile}{AB} : \underset{\smile}{EF}$ - - *Ch. III, Th. VII.*



Second: In the equal circles C and D, let ACB and EDF be incommensurable; then they will be proportional to $\underset{\smile}{AB}$ and $\underset{\smile}{EF}$; that is,

$ACB : EDF :: \underset{\smile}{AB} : \underset{\smile}{EF}$.

For, if the fourth term of the proportion is not equal to $\underset{\smile}{EF}$, it must be greater or less than $\underset{\smile}{EF}$. Suppose it to be greater, as $\underset{\smile}{EH}$: then

(1) $ACB : EDF :: \underset{\smile}{AB} : \underset{\smile}{EH}$.

Conceive $\underset{\smile}{AB}$ to be divided into equal parts, each of which is less than $\underset{\smile}{FH}$. Let one of these equal parts be applied, as a unit of measure, to $\underset{\smile}{EH}$, commencing at E. Between F and H there must be at least one point of division, as at K. $\underset{\smile}{AB}$ and $\underset{\smile}{EK}$ will then be commensurable; hence, A C B and E D K will be commensurable angles; hence,

(2) $ACB : EDK :: \underset{\smile}{AB} : \underset{\smile}{EK}$. From (1) and (2) we have

(3) $EDF : EDK :: \underset{\smile}{EH} : \underset{\smile}{EK}$ (*Ch. III, Th. VIII. Ex. 1.*)

But $EDF < EDK$; $\therefore \underset{\smile}{EH} < \underset{\smile}{EK}$ (*Ch. III, Th. IX*), which is impossible, since $\underset{\smile}{EH} > \underset{\smile}{EK}$.

Since the conclusion is false, the supposition which led to it is false; hence, the fourth term cannot be greater than $\underset{\smile}{EF}$.

By a process entirely similar, it may be shown that the fourth term cannot be less than $\underset{\smile}{EF}$.

Therefore, since the fourth term can be neither greater nor less than $\underset{\smile}{EF}$, it must be equal to $\underset{\smile}{EF}$, and

$ACB : EDF :: \underset{\smile}{AB} : \underset{\smile}{EF}$. *Q. E. D.*

Scholium 1.—If E D F be assumed as the unit of measure for angles, and its intercepted arc, $\underset{\smile}{EF}$, be assumed as the unit of measure for arcs, the above proportion becomes

$ACB : 1 :: \underset{\smile}{AB} : 1$, whence

$ACB = \underset{\smile}{AB}$; that is,

the value, or numerical measure, of a central angle is equal to that of its arc.

Scholium 2.—This is usually expressed, more briefly, by saying that *the central angle is measured by its intercepted arc.*

Scholium 3.—In accordance with this convenient method of measuring angles, the circumference is conceived as divided into

360 equal arcs, called *degrees.* Now, two diameters perpendicular to each other form four right angles at the centre, and divide the circumference into four equal arcs (*Th. XVII*), called *quadrants.*

Hence, the right angle is called an angle of 90 degrees, since it intercepts one-fourth of the circumference, which is 90 degrees. An arc of 45 degrees is the measure of half a right angle; an arc of 1 degree ($\frac{1}{90}$ of a quadrant) is the measure of $\frac{1}{90}$ of a right angle, which is hence called an angle of 1 degree. Thus, from the number of degrees that measure it, we have a definite idea of the size of the angle; and we may understand the conventional statement, that an angle has the same number of degrees as its intercepted arc.

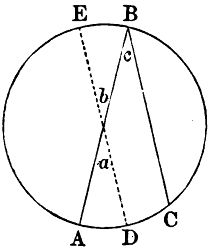
Scholium 4.—The arc of 1 degree (1°) is further subdivided into parts called *minutes* and *seconds;* the minute being $\frac{1}{60}$ of a degree, and written 1′; the second being $\frac{1}{60}$ of a minute, and written 1″.

Scholium 5.—In the comparison of angles with each other, it is to be observed that the measuring arcs must be described with *equal* radii.

THEOREM XX.

An inscribed angle is measured by one-half its intercepted arc.

There may be three cases: one side may be a diameter; the centre of the circle may lie within the angle; the centre may lie without the angle.



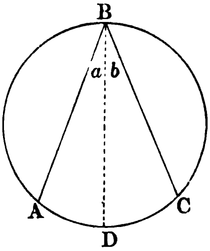
First: Let c be an inscribed angle, and A B a diameter; then will $c = \frac{1}{2}\,\underset{\smile}{\mathrm{A\,C}}$.

For, draw the diameter, E D, parallel to B C.

$$a = b;\quad \therefore\quad \underset{\smile}{\mathrm{A\,D}} = \underset{\smile}{\mathrm{E\,B}};$$

but $\underset{\smile}{\mathrm{D\,C}} = \underset{\smile}{\mathrm{E\,B}}$ (*Th. XII*); $\therefore\ \underset{\smile}{\mathrm{A\,D}} = \underset{\smile}{\mathrm{D\,C}}$;

$\therefore\ \underset{\smile}{\mathrm{A\,D}} = \frac{1}{2}\,\underset{\smile}{\mathrm{A\,C}}$; but $a = \underset{\smile}{\mathrm{A\,D}}$ (*Th. XIX*, *S. 1*);
$\therefore\ a = \frac{1}{2}\,\underset{\smile}{\mathrm{A\,C}}$; but $a = c$;
$\therefore\ c = \frac{1}{2}\,\underset{\smile}{\mathrm{A\,C}}$.



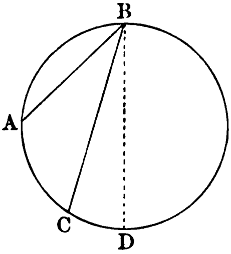
Second: Let the centre lie within the angle; then will $\mathrm{A\,B\,C} = \frac{1}{2}\,\underset{\smile}{\mathrm{A\,C}}$.

For, draw the diameter B D.

By the preceding case, $a = \frac{1}{2}\,\underset{\smile}{\mathrm{A\,D}}$, and $b = \frac{1}{2}\,\underset{\smile}{\mathrm{D\,C}}$;

$\therefore\ a + b = \frac{1}{2}\,\underset{\smile}{\mathrm{A\,D}} + \frac{1}{2}\,\underset{\smile}{\mathrm{D\,C}} = \frac{1}{2}\,(\underset{\smile}{\mathrm{A\,D}} + \underset{\smile}{\mathrm{D\,C}}) = \frac{1}{2}\,\underset{\smile}{\mathrm{A\,C}}$: but $a + b = \mathrm{A\,B\,C}$;

$\therefore\ \mathrm{A\,B\,C} = \frac{1}{2}\,\underset{\smile}{\mathrm{A\,C}}$.



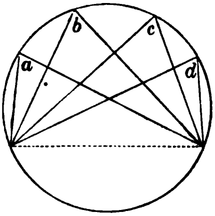
Third: Let the centre lie without the angle; then will $\mathrm{A\,B\,C} = \frac{1}{2}\,\underset{\smile}{\mathrm{A\,C}}$.

For, draw the diameter B D.

By the preceding cases we have

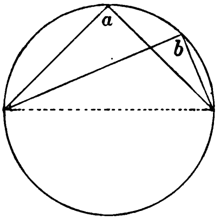
(1) $\mathrm{A\,B\,D} = \frac{1}{2}\,\underset{\smile}{\mathrm{A\,D}}$, and
(2) $\mathrm{C\,B\,D} = \frac{1}{2}\,\underset{\smile}{\mathrm{C\,D}}$.

(1) − (2) = (3) $\mathrm{A\,B\,D} - \mathrm{C\,B\,D} = \frac{1}{2}\,\underset{\smile}{\mathrm{A\,D}} - \frac{1}{2}\,\underset{\smile}{\mathrm{C\,D}} = \frac{1}{2}\,(\underset{\smile}{\mathrm{A\,D}} - \underset{\smile}{\mathrm{C\,D}}) = \frac{1}{2}\,\underset{\smile}{\mathrm{A\,C}}$: but $\mathrm{A\,B\,D} - \mathrm{C\,B\,D} = \mathrm{A\,B\,C}$; $\therefore\ \mathrm{A\,B\,C} = \frac{1}{2}\,\underset{\smile}{\mathrm{A\,C}}$. *Q. E. D.*



Cor. I.—*All the angles inscribed in the same segment are equal.*

For, each is measured by half the same arc.



Cor. II.—*Any angle inscribed in a semicircle is a right angle.*

For, it is measured by one-half of a semicircumference.

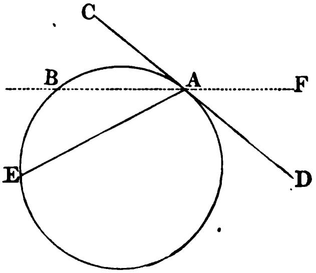
Cor. III.—*A central angle is twice the inscribed angle which intercepts the same arc.*

QUERIES.

1. In the foregoing Theorem, what is given,—what is the hypothesis?
2. What is required to be proved?
3. What are the leading steps, or essential points, in the proof of the first case?
4. Why does $a = b$?
5. Why does $\overset{\frown}{EB} = \underset{\smile}{AD}$?
6. Why does $a = c$?
7. What axioms are involved in the three cases?

THEOREM XXI.

An angle formed by a tangent and a chord is measured by one-half the intercepted arc.

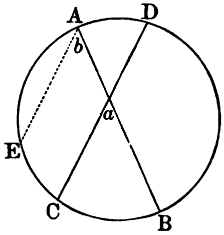


Let C D be a tangent and A E a chord; then will $CAE = \frac{1}{2}\overset{\frown}{AE}$.

For, draw the secant B F, and suppose it to revolve about A as a centre. As it revolves, the second point of intersection, B, will approach A; and when it reaches A, B F will become C D, E B will become E A, and B A E will become C A E; but in every position of B F, B A E is measured by half its intercepted arc (*Th. XX*); hence, it will be so measured when B F becomes a tangent; hence $CAE = \frac{1}{2}\overset{\frown}{AE}$. *Q. E. D.*

THEOREM XXII.

An angle formed by two intersecting chords is measured by one-half the sum of the arcs intercepted between its sides and the sides of its vertical angle.

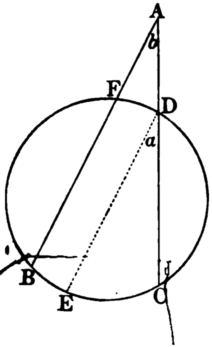


Let a be an angle formed by the two intersecting chords, A B and C D; then will $a = \frac{1}{2}(\overset{\frown}{AD} + \underset{\smile}{CB})$.

For, draw the chord A E parallel to C D. Now, $b = \frac{1}{2}(\underset{\smile}{EC} + \underset{\smile}{CB})$; but $\underset{\smile}{EC} = \overset{\frown}{AD}$ (why?); hence, $b = \frac{1}{2}(\overset{\frown}{AD} + \underset{\smile}{CB})$; but $b = a$; hence, $a = \frac{1}{2}(\overset{\frown}{AD} + \underset{\smile}{CB})$. *Q. E. D.*

THEOREM XXIII.

An angle formed by two secants intersecting without the circle is measured by one-half the difference of the intercepted arcs.

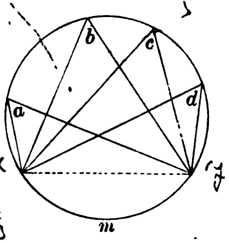


Let b be an angle formed by two secants intersecting without the circle; then will $b = \frac{1}{2}(\underset{\smile}{BC} - \overset{\frown}{DF})$.

For, draw D E parallel to A B. Now, $a = \frac{1}{2}\underset{\smile}{EC} = \frac{1}{2}(\underset{\smile}{BC} - \underset{\smile}{BE})$; but $\underset{\smile}{BE} = \overset{\frown}{DF}$ (why?); hence, $a = \frac{1}{2}(\underset{\smile}{BC} - \overset{\frown}{DF})$; but $a = b$ (why?); hence, $b = \frac{1}{2}(\underset{\smile}{BC} - \overset{\frown}{DF})$.

Q. E. D.

EXERCISES.



1. Which of the angles a, b, c, d, is the largest?

2. Are a and c inscribed in the same segment?

3. How is an inscribed angle measured?

4. Any angle of an inscribed equilateral triangle intercepts what part of the circumference?

5. By what part of a circumference is any angle of an inscribed equiangular hexagon measured? / 6

6. The vertical angle of an inscribed isosceles triangle is ⅓ of a right angle. What part of the circumference does either of its equal angles intercept? / 5

7. Two of the opposite sides of a quadrilateral are each 15 in length, and parallel. What is the kind of figure?

Suggestion.—15 is the number of times that either of the given sides contains the unit of length, which may be assumed at pleasure.

8. If two of the opposite sides were 15 each, and the other two were 10 each, what would be the form of the figure? Would these facts determine the angles?

9. How many degrees in an angle inscribed in ¾ of a circle?

10. Two secants, drawn from a point without the circle, intercept arcs which are $\frac{1}{18}$ and $\frac{1}{6}$ of the circumference. How many degrees in the angle?

11. The parallelogram whose diagonals are equal is a rectangle.

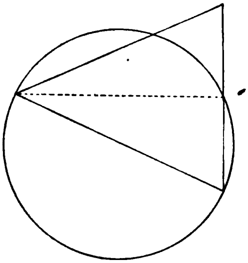
12. If from any point in the base of an isosceles triangle perpendiculars

be drawn to the other two sides, their sum is constant, and equal to the perpendicular from one of the equal angles to the opposite side.

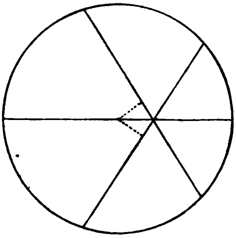
Suggestion.—Construct an isosceles triangle having such parts as the conditions require, namely, two perpendiculars from a point in the base to the sides, and a third perpendicular from either of the equal angles to the opposite side. State the theorem with reference to the diagram. To the third perpendicular draw, from the common point in the base, a fourth. The demonstration is based mainly upon the equality of triangles.

13. All the equal chords in a circle may be touched by another circle. (*See Th. III.*)

14. If two opposite sides of an inscribed quadrilateral are equal, the other two sides are parallel. (*See Ths. IV and V.*)

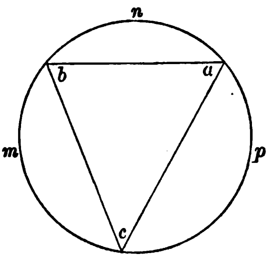


15. If the diameter of a circle be made one of the equal sides of an isosceles triangle, the base will be bisected by the circumference.



16. Two chords of a circle which cut a diameter in the same point and make equal angles with it, are equal.

17. An inscribed angle is acute or obtuse, according as it is inscribed in a segment greater or less than a semicircle.



18. The sum of the angles of a triangle equals two right angles. (*See Th. IX.*)

19. An equiangular triangle is equilateral; and, conversely, an equilateral triangle is equiangular. (*See Ths. IV, V and XX.*)

20. In any triangle the greater side lies opposite the greater angle, and conversely.

CHAPTER V.

CONSTRUCTIONS.

OBSERVATIONS.

The student may be surprised that methods for constructing diagrams have not hitherto been given. But, first, it was unnecessary that our diagrams should be rigorously exact, since they were employed merely to aid our mental conceptions in reasoning about *principles;* and, secondly, their precise construction involves the principles previously discussed.

The constructions of Elementary Geometry are effected by the straight line and circumference, with the aid of a ruler and compass.

The object of these constructions is twofold,—to afford an excellent review of geometrical facts, and to impart skill in the application of knowledge.

While they may be understood by simple inspection, the student is warned against a method of study so partial and unsatisfactory, unattended by either the profit or the delight of art. They will be impressed upon the mind with far greater ease and force, if, in every instance, they are *actually drawn.* Science is acquired by study—art by practice.

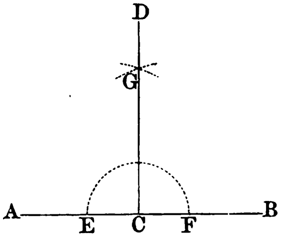
It will be observed that the solution of a problem consists of three parts:

1. *To draw the parts given;*
2. *To draw the parts requisite for construction;*
3. *To give the reasons of the process, or, the proof that the construction is correct.*

Note.—‘Q. E. F.’ stands for *quod erat faciendum*, which is read, *which was to be done.*

PROBLEM I.

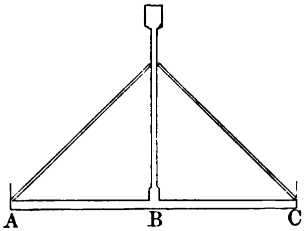
At a given point within a line, to erect a perpendicular to that line.



Let A B be the given line, and C the given point within it.

From C as a centre, with any radius, describe an arc intersecting A B in the points E and F; from E and F as centres, with a radius greater than one-half of E F, describe arcs intersecting in G. Through G draw D C, which will be the required perpendicular.

For, by construction, C is one point equally distant from E and F, and G is another; hence D C is perpendicular to E F, and therefore to A B. (*Ch. II, Th. XX, Ex. 2.*) *Q. E. F.*



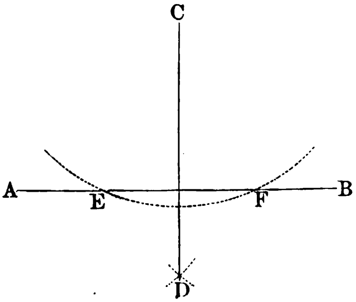
Note.—Suppose a carpenter wished to mortise a 'king-post' into a 'purline' at a given point B, and at right angles to it, without the use of a set-square.

Having laid off, on the 'purline,' B A = B C, he might take two poles of equal length, and, fastening their lower extremities at A and C, place their upper extremities against the 'post'; then move the post, till the ends of the poles are just opposite, or in the same straight line, and the 'king-post' will be perpendicular to the 'purline.'

What principle guarantees the accuracy of this method?

PROBLEM II.

From a given point without a line, to draw a perpendicular to that line.

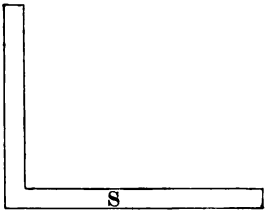


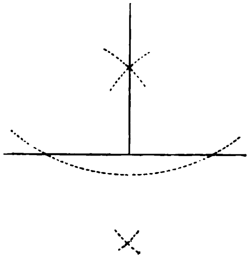
Let A B be the given line, and C the given point.

With C as a centre, and a radius that will cut A B, describe an arc intersecting A B in the points E and F; from E and F as centres, with radius greater than half of E F, decribe arcs intersecting in D. Draw C D, which will be the perpendicular required.

For, by construction, C is one point equally distant from E and F, and D is another; hence, C D is perpendicular to E F, and therefore to A B. (*Ch. II, Th. XX, Ex. 2.*) *Q. E. F.*

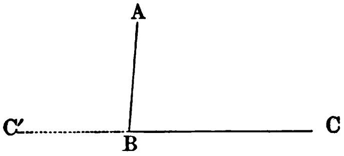
Note.—Comparatively few of the so-called carpenters' squares are rigorously true; and those that are so, are liable to lose their precision. Now, in the first place, either of the preceding problems enables us to delineate an exact right angle, by which such an instrument may be constructed with the utmost precision. In the second place, either of these problems indicates the method of testing the accuracy of such an instrument.





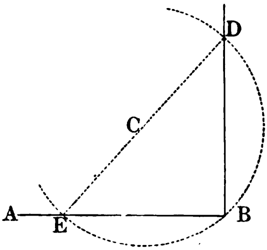
For, we have only accurately to construct one line perpendicular to another, and then apply the square to the right angle thus formed. If its two edges coincide with the sides of the right angle constructed, it is precise; otherwise, it is faulty.

Or thus: Place the instrument upon paper, and with a pencil trace its angle, as A B C. Now produce one side, as B C; and apply the instrument to the angle A B C′: if A B C is a right angle, A B C′ must be a right angle also (*Ch. II, Th. I*), and the instrument will coincide with it, and therefore be correct; otherwise, it is faulty.



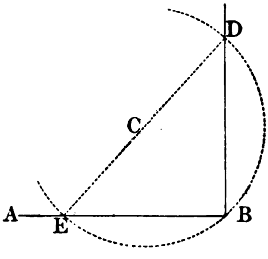
PROBLEM III.

At a given extremity of a line, to erect a perpendicular to that line.



Let A B be the given line, and B the given extremity.

Assume any point, C, outside of the given line, and from C as a centre, with a radius equal to the distance from C to B, describe an arc intersecting A B, as in E; through C, draw the diameter



E D; join D and B. B D is the required perpendicular.

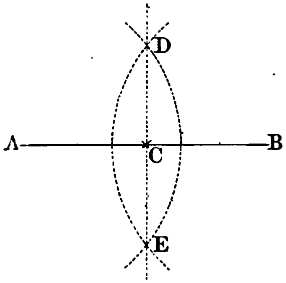
For, the angle E B D is inscribed in a semicircle, and is therefore a right angle; hence, D B is perpendicular to A B at the given extremity. *Q. E. F.*

Note.—By this process, the end of a board may be squared without a carpenters' square.

In squaring the foundations of a building, or in laying out a garden, the rod and chalk-line may be used.

PROBLEM IV.

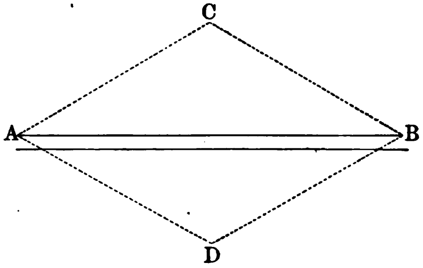
To bisect a given straight line.



Let A B be the given line.

From A as a centre, with a radius greater than the half of A B, describe an arc; from B as a centre, with the same radius, describe a second arc intersecting the first in D and E; join D and E. D E is the required bisector.

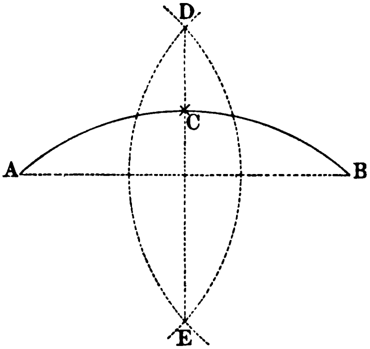
For, D and E are points equally distant from the extremities A and B; hence D E is perpendicular to A B at its middle point. *Q. E. F.*



Note.—A gardener wishing to construct a walk at right angles to another at its middle point, may, for instance, fasten the ends of a chalk-line, longer than the given walk, as at A and B; then taking the line at its middle point, carry it back on either side of A B, till it is stretched, marking the remotest points of tension, as C and D, by pins in the ground: he then stretches his line from C to D, which gives the direction and position of the path sought.

PROBLEM V.

To bisect a given arc.



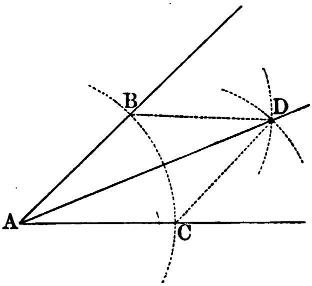
Let $\overset{\frown}{AB}$ be the given arc.

Draw the chord A B, and bisect it, as in Prob. IV. The bisector of the chord will bisect the arc. (*Ch. IV, Th. II, Cor. I.*) *Q. E. F.*

[Let the student give the *process* in full, and the *reasons.*]

PROBLEM VI.

To bisect a given angle.



Let A be the given angle.

From A as a centre, with any radius, describe an arc cutting the sides of the angle in B and C; from B and C as centres, with radii equal and sufficiently great, describe arcs intersecting in D. Draw A D, which will be the required bisector.

For, the triangles A B D and A C D are mutually equilateral (why?), and therefore equiangular; hence C A D = B A D.

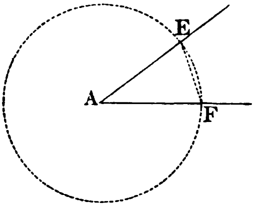
Q. E. F.

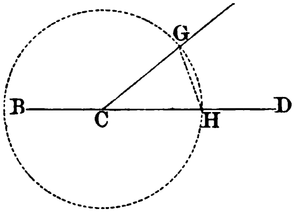
Scholium.—Similarly, each of the halves could be bisected, and the bisections so continued to any desired extent. The description of the arc from A is merely a convenient method of making A B = A C.

PROBLEM VII.

At a given point in a straight line, to construct an angle equal to a given angle.

Let A be the given angle, and C the given point in the given line.





From A as a centre, with any radius, describe an arc cutting the sides in E and F; from C as a centre, with the same radius, describe an arc cutting B D in H; from H as a centre, with a radius equal to the chord E F, describe a third arc intersecting the second in G; draw C G. G C H is the required angle.

For, $EF = GH$;

$\therefore \overset{\frown}{EF} = \overset{\frown}{GH}$;

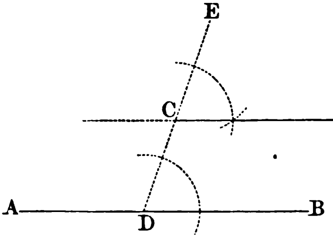
$\therefore GCH = A$. *Q. E. F.*

PROBLEM VIII.

Through a given point, to draw a line parallel to a given line.

Let C be the given point, and A B the given line.

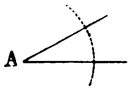
Through C draw any straight line, as E D; at the point C, in this line, construct an angle, E C F, equal to E D B (*Prob. VII*).

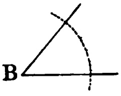


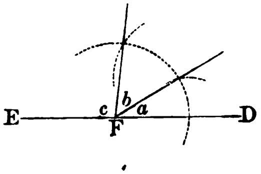
Because the opposite exterior and interior angles, E C F and C D B, are equal, C F is parallel to A B. (*Ch. II, Th. IX, Ex.*) *Q. E. F.*

PROBLEM IX.

Two angles of a triangle being given, to find the third.







Let A and B be the given angles of the triangle, and let C denote the third.

Draw an indefinite line, E D; and, at any point in it, as F, make $a = A$, and $b = B$. c is the required angle.

For, $A + B + C = 2R$ (*Ch. II, Th. XII*), and

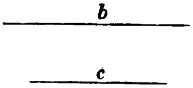
$a + b + c = 2R$ (*Ch. II, Th. I, Cor.*); hence,

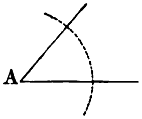
$a + b + c = A + B + C$: but, by construction,

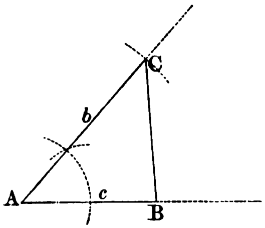
$a + b = A + B$; $\therefore c = C$. *Q. E. F.*

PROBLEM X.

Two sides of a triangle and their included angle being given, to construct the triangle.







Let b and c be the given sides, and A their included angle.

At any point, A, in a straight line, construct an angle equal to the given angle; on the sides lay off distances equal to the given sides, as A C and A B; draw B C. A B C is the required triangle.

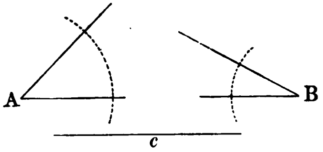
For, all triangles constructed with the given data must be equal, since they would have two sides and the included angle of the one equal to two sides and the included angle of the other.

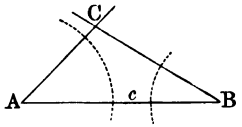
Q. E. F.

PROBLEM XI.

One side of a triangle and its adjacent angles being given, to construct the triangle.

Let c be the given side, and A and B its adjacent angles.





Draw a line A B $= c$, and at its extremities construct angles equal to the given angles; produce the sides of these angles till they intersect in C. A B C is the required triangle.

For, all triangles constructed with these data must be equal. (*Ch. II, Th. V.*) *Q. E. F.*

Scholium.—If simply one side and two angles are given, one being opposite, and the other adjacent, to the given side, the third angle may be found by Prob. IX, and thus we shall always have given one side and its adjacent angles.

QUERIES.

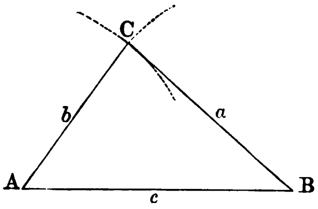
1. If the two given angles were obtuse, could the triangle be constructed?

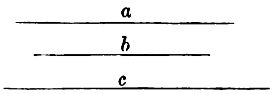
2. If one were obtuse and the other acute, but their sum equal to two right angles?

PROBLEM XII.

The three sides of a triangle being given, to construct the triangle.

Let a, b, and c be the given sides.





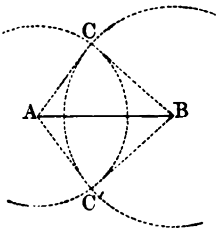
Draw A B $= c$*;* from A as a centre, with a radius equal to b, describe an arc; from B as centre, with a radius equal to a, describe a second arc intersecting the first in C; join the point of intersection with A and B. A B C is the required triangle.

For, all triangles constructed with these data must be equal. (*Ch. II, Th. XVII.*) *Q. E. F.*

QUERIES.

1. If $a + b$ were less than c, would the arcs in the figure intersect?
2. If $a + b = c$?
3. If $a < c - b$?
4. What conditions must the data fulfil?

Note.—This problem finely illustrates the intersection of *loci.* The point sought for is the vertex C, which must evidently satisfy two conditions,—it must be at a distance b from A, and at the same time at a distance a from B. Now the locus of points at the distance b from A, is the circumference described from A as centre, with radius b*;* and the locus of points at the distance a from B, is the circumference described from centre B, with radius a. Hence, the intersections of these loci will determine the position of the vertex, and the triangle can have either of two positions, A B C, or A B C′.

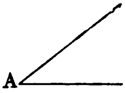


PROBLEM XIII.

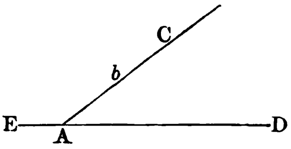
Two sides of a triangle and an angle opposite one of them being given, to construct the triangle.

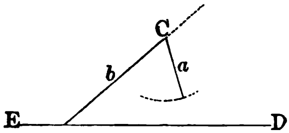
1 *When the given angle is acute:*

Let A be the given angle.

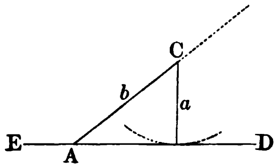


Draw an indefinite line, and at any point, as A, construct an angle equal to the given angle; take A C = to the given adjacent side, which we will denote by b; denote the side opposite A by a, and from C as a centre, with a as a radius, describe an arc. Then:

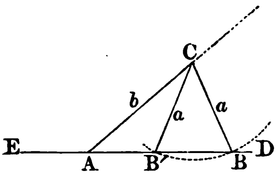




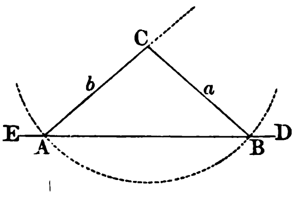
(1) If a is *less* than the perpendicular distance from C to E D, there will be *no* point of intersection or contact, and the triangle, with such data, cannot be constructed.



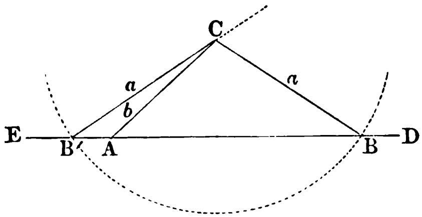
(2) If a *equals* the perpendicular from C to E D, there will be no point of intersection, but one of contact, and the triangle formed by joining this point of contact with C will be *right-angled* (*Ch. IV, Th. XI, Cor.*).



(3) If a is *greater* than the perpendicular, yet less than b, the described arc will *cut* E D in two points, as B and B', on the *same* side of A, and the required triangle may be either A B C or A B' C; whence this case is properly called *ambiguous*.

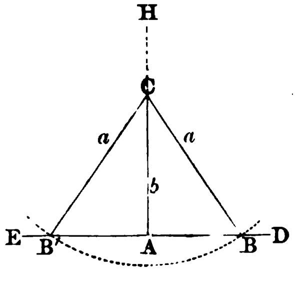


(4) If $a = b$, the described arc will cut E D in two points, as before, but one point of intersection will be at A; and as B' of the preceding case coincides with A, the triangle A B' C will vanish into the straight line A C, the remaining triangle becoming *isosceles*.



(5) If $a > b$, the described arc will cut E D in two points, on *opposite* sides of A, and two triangles may be formed by joining the points of intersection with C; but only one of these will fulfil the conditions,—A B C; for, though the triangle A B' C has the two given sides, yet the angle opposite a is obtuse, whereas our data require it to be *acute*.

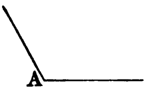
2. *When the given angle is right:*

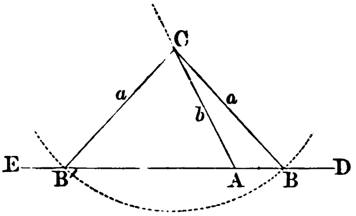


Let A be the given angle.

Draw E D, and at any point, A, construct H A B = A; take A C equal to the given adjacent side, *b*. Now, the required triangle will be right-angled; hence, the describing radius (which is the side opposite the given angle) must be greater than *b*; hence, the arc described will cut E D in two points on opposite sides of A, and two triangles may be formed, A B C and A B′ C, both of which conform to the data; but, as they are *equal in all their parts*, there is, in reality, with these data, but *one solution.*

3. *When the given angle is obtuse:*





Let A be the given angle.

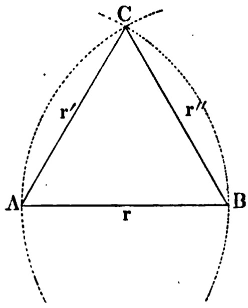
At any point, as A, in the line E D, construct an angle equal to the given angle; take A C = *b*. In the required triangle, the side opposite the given angle must be the greatest (why?); hence, the describing radius will be greater than *b*, and two triangles may be formed, as A B C and A B′ C; but only one of them, namely, A B C, conforms to the data; for, in the triangle A B′ C, the angle opposite *a* is acute, whereas, the data require it to be obtuse. *Q. E. F.*

QUERIES.

1. In (4) and (5), why is *a* greater than the perpendicular from C to E D?
2. In A B′ C of (5), why must the angle opposite *a* be obtuse?
3. In A B′ C, of '2,' why is C A B′ a right angle?
4. Why must *a*, in '3,' be greater than *b*?

PROBLEM XIV.

On a given base, to construct an equilateral triangle.



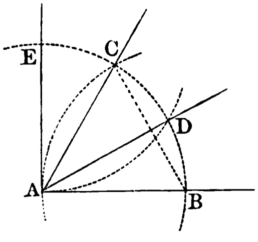
Let A B be the given base.

From A as centre, with radius A B, describe an arc; from B as centre, with same radius, describe a second arc intersecting the first in C; draw A C and B C. A B C is the required triangle.

For, $r' = r$, being radii of the circle whose centre is A; and $r'' = r$, being radii of the circle whose centre is B; $\therefore$ $r'' = r' = r$; hence, A B C is equilateral. *Q. E. F.*

PROBLEM XV.

To trisect a right angle.



Let A be the given angle.

From centre A, with any assumed radius, describe an arc cutting the sides in E and B; on A B construct an equilateral triangle.

Angle C A B = 60°; hence,

E A C = 30°.

Now, bisect C A B (*Prob. VI*). Otherwise, on A E construct an equilateral triangle; then E A D = 60°;

$\therefore$ D A B = 30°;

but, also, E A C = 30°;

$\therefore$ C A D = 30°.

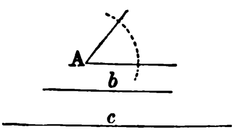
$\therefore$ E A C = C A D = D A B.

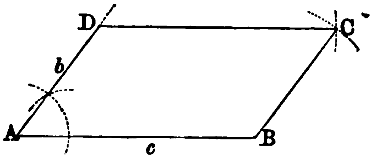
Q. E. F.

PROBLEM XVI.

Two sides and the included angle of a parallelogram being given, to construct the parallelogram.

Let A be the given angle, b and c the given sides.





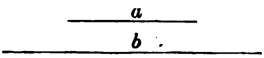
Draw A B = c, and at A construct an angle equal to the given angle; lay off A D = b; from centre D, with radius c, describe an arc; from centre B, with radius b, describe a second arc intersecting the first in C. A B C D is the required parallelogram.

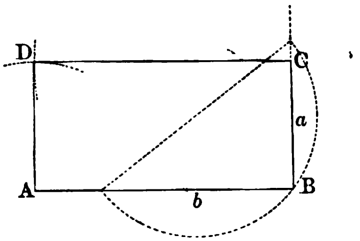
For, its opposite sides are equal, and it is constructed with the data. *Q. E. F.*

PROBLEM XVII.

Two adjacent sides of a rectangle being given, to construct the rectangle.

Let a and b be the given sides.





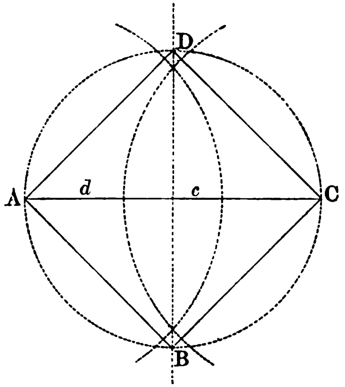
Draw A B = b, and at B erect a perpendicular (*Prob. III*); take B C = a; from centre C, with radius b, describe an arc; from centre A, with radius a, describe a second arc intersecting the first in D; draw A D and C D. A B C D is the required rectangle.

For, its opposite sides are equal, its angles are right angles, and it is constructed with the data. *Q. E. F.*

Otherwise, through C, draw C D parallel to B A; and through A, A D parallel to B C (*Prob. VIII*).

PROBLEM XVIII.

The diagonal of a square being given, to construct the square.



Let *d* be the given diagonal.

Bisect the given diagonal (*Prob. IV*) in *c*, and from centre *c*, with radius A *c*, describe a circle cutting the bisector in D and B; join D and B with A and C. A B C D is the required square.

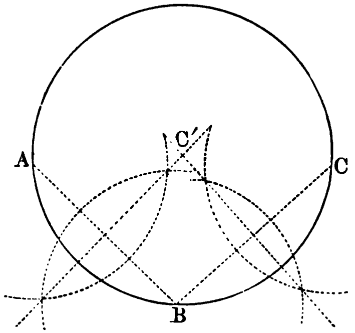
For, $\overset{\frown}{AD} = \overset{\frown}{DC} = \overset{\frown}{CB} = \overset{\frown}{BA} = 90°$;

∴ A D = D C = C B = B A; hence, A B C D is equilateral; moreover, each of its angles is inscribed in a semicircle, and is therefore a right angle; hence, A B C D is right-angled; hence, a square, and is constructed with the given diagonal.

Q. E. F.

PROBLEM XIX.

To find the centre of a given circle.

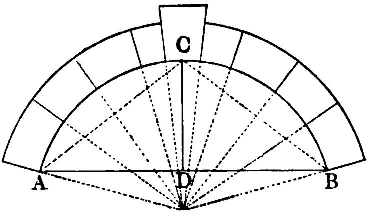


Let C′ be the given circle.

Take any three points in the circumference, and bisect the chords A B and B C joining them (*Prob. IV*). These bisectors, being perpendicular to A B and B C, each pass through the centre of the circle; hence, their point of intersection, C′, is the required centre. *Q. E. F.*

Scholium.—The same construction,—assuming three points and bisecting their subtending chords by perpendiculars,—serves to find the centre of a given arc; also to pass a circumference through the vertices of a triangle, or through any three points not in the same straight line.

Note.—This problem will be found serviceable in the construction of a circular arch, when the 'span,' A B, and 'rise,' C D, are given, and it is required to find the curve. The *voussoirs*, or arch-stones, are so shaped that the joints between them are continuations of the radii. If not convenient to use the centre, how might the joints be found? A normal at the point of tangency does what?

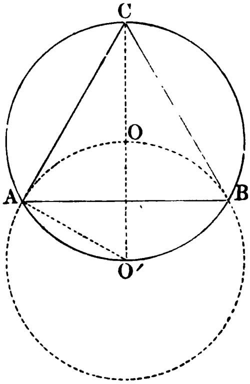


PROBLEM XX.

In a given circle to inscribe an equilateral triangle.

Let O be the given circle.

Draw the diameter C O′; from O′ as centre, with radius equal to radius of given circle, describe a second circle cutting the first in A and B; draw A B, B C, A C. A B C is the required triangle.



For, draw A O′. C A O′ is a right angle; hence, A C is tangent to circle O′ at A (*Ch. IV, Th. XI*); and, similarly, B C at B; hence A C = B C.

$\therefore A = \frac{1}{2} AOB$, and

$B = \frac{1}{2} AOB$ - - - - - - *Ch. IV, Th. XXI*;

$\therefore A = B$

$C = \frac{1}{2} AO'B$ - - - - - - *Ch. IV, Th. XX*;

But $AO'B = AOB$ - - - - - - *Ch. IV, Th. XIII*;

$\therefore \frac{1}{2} AO'B = \frac{1}{2} AOB$;

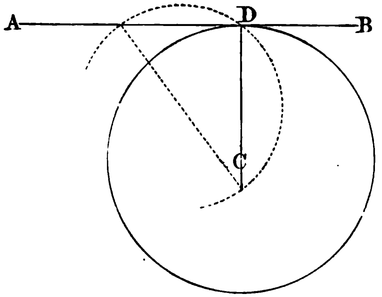
$\therefore C = A = B$.

Hence, the triangle is equiangular, and therefore equilateral.

Q. E. F.

PROBLEM XXI.

To draw a tangent to a given circle at a given point of the circumference.



Let C be the given circle, and D the given point.

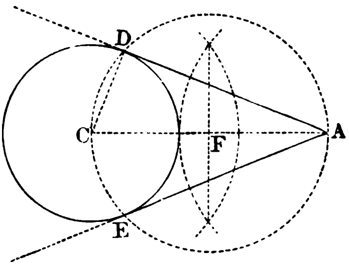
To the given point draw a radius C D; at D erect a perpendicular A B (*Prob. III*). A B is the required tangent (*Ch. IV, Th. XI*).

Q. E. F.

Scholium.—If the centre of the circle is not given, it may be found by *Prob. XIX.*

PROBLEM XXII.

To draw a tangent to a circle from a given point without the circle.



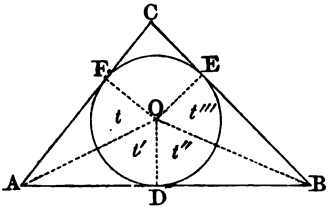
Let C be the given circle, and A the given point without.

Join the given point with the centre; bisect A C; from centre F, with radius = C F, describe a circle cutting the given circle in points D and E; join the points of intersection with A. The required tangent will be either A D or A E.

For, A D C is a right angle (*Ch. IV, Th. XX, Ex. 2*); hence, A D is perpendicular to C D, a radius of the given circle, at its extremity; hence, A D is tangent to circle C (*Ch. IV, Th. XI*); similarly, A E is a second tangent. *Q. E. F.*

PROBLEM XXIII.

To inscribe a circle in a given triangle.



Let A B C be the given triangle.

Bisect any two angles, as A and B (*Prob. VI*), producing their bisectors till they meet in O; from O to the sides draw the perpendiculars O D, O E, and O F.

Now, the triangles t and t', being mutually equiangular, are equal. For like reason, $t'' = t'''$.

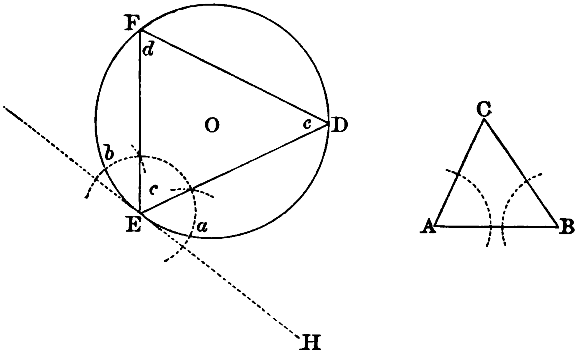
$\therefore$ (1) O F = O D, and
(2) O E = O D;
$\therefore$ (3) O F = O E = O D.

Therefore, a circle described from O as centre, with radius = O D, will touch the sides of the triangle at D, E, and F, and the circle, by definition, will be inscribed in the given triangle.

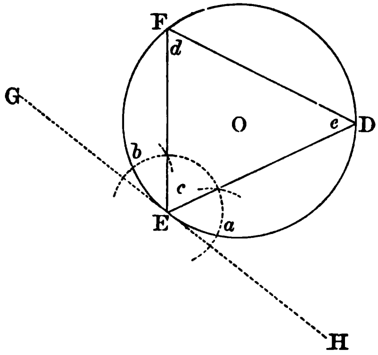
PROBLEM XXIV.

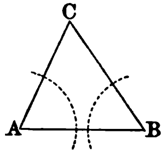
In a given circle, to inscribe a triangle equiangular to a given triangle.

Let O be the given circle, and A B C the given triangle.



Draw a tangent to the circle at any point, as E; at E, make angle a = A, producing its side to meet the circumference in D; at the same point, make $b = \mathrm{B}$, producing its side to meet the circumference in F; join D and F. E D F is the required triangle.





For, by construction, $a + b = \mathrm{A} + \mathrm{B}$;

$\therefore$ (1) $c = \mathrm{C}$. $a = \frac{1}{2}\,\overset{\frown}{\mathrm{ED}}$;

but $d = \frac{1}{2}\,\overset{\frown}{\mathrm{ED}}$; $\therefore$ $d = a$; but $a = \mathrm{A}$;

$\therefore$ (2) $d = \mathrm{A}$. Again, $b = \frac{1}{2}\,\overset{\frown}{\mathrm{EF}}$; but

$e = \frac{1}{2}\,\overset{\frown}{\mathrm{EF}}$; $\therefore$ $e = b$; but $b = \mathrm{B}$;

$\therefore$ (3) $e = \mathrm{B}$. *Q. E. F.*

EXERCISES.

1. To construct the complement of a given angle. (*See Prob. I or III.*)

2. To construct the supplement of a given angle. (*See Ch. II, Th. I.*)

3. At a given point in a given line, to construct an angle of 45°. (*See Probs. I and VI.*)

4. At a given point in a given straight line, to construct an angle of 30°. (*See Prob. XV.*)

5. On a given base, to construct an isosceles triangle of which the vertical angle shall be 90°.

Suggestion.—The angles at the base equal what?

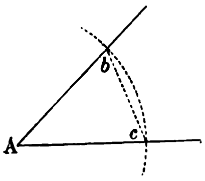
6. To construct a triangle on a given base whose adjacent angles shall be 60° and $22\frac{1}{2}$°, respectively.

7. To bisect an angle by Th. II, Ch. IV.

8. On a given base, to construct a square. (*See Prob. III.*)

9. On a given base, with a given vertical angle, to construct an isosceles triangle.

Suggestion.—Let A be the given angle. At the extremities of the given base, construct angles = *b* and *c*.



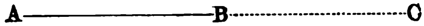
10. In a given circle, to inscribe a triangle that shall be equiangular to a given triangle, and also similar to it in position.

Suggestion.—The principles involved are the same as in Prob. XXIV.

11. To trisect one-twelfth of a right angle.

Suggestion.—First trisect a right angle, to get one-third.

12. On a given line, to construct an isosceles triangle whose equal sides shall each be twice the base.



Suggestion.—Let A B be the given line.

Produce it to C, making B C = A B; from centres A and B, with radius = A C, *etc.*

13. The perpendicular from the vertex of the vertical angle to the base being given, to construct an equilateral triangle.

Suggestion.—At the extremity B erect an indefinite perpendicular; at A, on each side of A B, construct, *etc.*

14. To describe a circle that shall touch two given straight lines, not parallel, touching one of them at a given point.

Suggestion.—Produce the lines till they intersect; bisect their angle, and at the given point erect a perpendicular.

15. To divide a circumference into two arcs, one of which shall be twice the other.

Suggestion.—Inscribe in the circle an equilateral triangle.

16. To construct a rhombus, the length of the two diagonals being given.

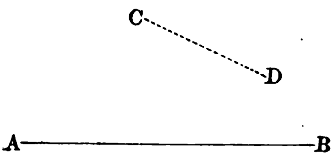
17. The hypotenuse and one of the acute angles of a right triangle being given, to construct the triangle.

18. The hypotenuse and one side being given, to construct the triangle.

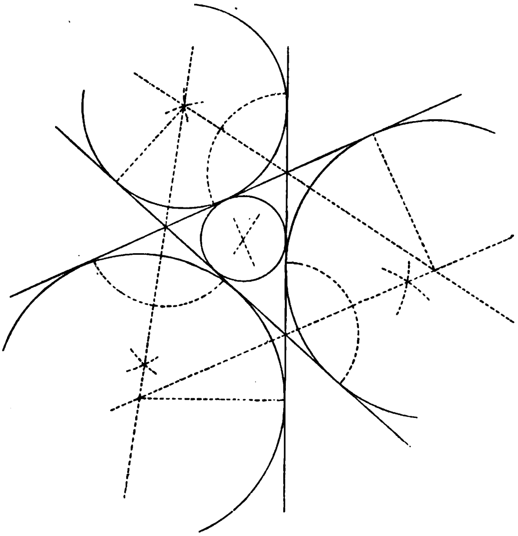
19. To find a point in a given straight line equidistant from two given points.

Suggestion.—Let C and D be the given points and A B the given line; bisect, etc.

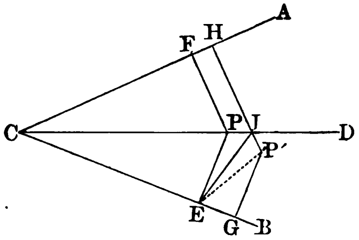
What limitation to this problem?



20. To describe four circles that shall be tangent to three intersecting straight lines.



21. To find the locus of the points of bisection of the equal chords of a circle. (*See Theorems on equal chords, Ch. III.*)



22. Prove that the bisector of an angle is the complete locus of points equally distant from the sides of the angle.

Suggestion.—Let P be any point on the bisector, and P′ any point not in the bisector: to prove $PE = PF$, and $P'G < P'H$. (*See Ch. II, Ths. III, V, and XIX.*)

23. To find the locus of the vertices of all the right angles whose sides pass through two given points.

24. To find the locus of the points, the sum of whose distances from two intersecting straight lines is constant.

25. About a given circle to circumscribe an equilateral triangle.

Suggestion.—Draw three radii making angles with each other of 120°; at their extremities construct tangents.

CHAPTER VI.

EQUALITY AND MEASUREMENT OF POLYGONS.

DEFINITIONS.

1. When used with reference to figures, the word *equal* and the sign = mean simply equality of area.

We have hitherto had instances of figures that could be made to coincide, and that were shown to be equal by such coincidence. Such figures are congruently or identically equal.

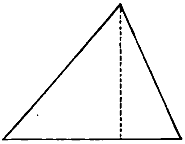
But non-congruent figures, which cannot be made to coincide, may nevertheless be equal in area. Thus, a triangular field may be as large as a rectangular one. Magnitudes having the same size, but not superposable, are sometimes called *equivalent*.

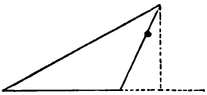
2. The **Area** of a surface is its numerical measure; that is, the number of times it contains *some other surface* taken as a **unit**.

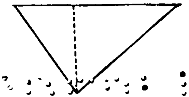
The unit of surface usually assumed is the **square**, whose side is some assumed unit of length; as, a *square inch*, a *square foot*, or a *square yard;* meaning a square one inch on a side, one foot on a side, one yard on a side.

Thus, the area of a field is the number of squares, 1 foot or 1 rod on a side, which it would take to cover it.

3. By the 'rectangle of two lines' is meant the 'product of two lines'; and the 'product of two lines' must be understood, not as the product of the lines themselves, but as the product of their *numerical measures.* Only the numbers representing the lines can be multiplied.







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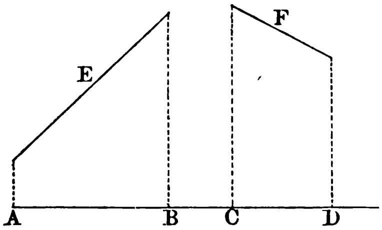
4. The **Altitude of a triangle** is the perpendicular distance from the vertex of either angle to the opposite side, or to that side produced.

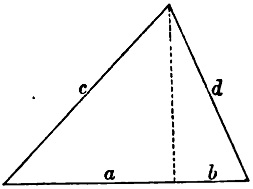
5. The **Altitude of a parallelogram** is the perpendicular distance between two opposite sides; one of which is then called the *upper* base, and the other the *lower* base.

6. The **Altitude of a trapezoid** is the perpendicular distance between its parallel sides, one of which is the *upper*, and the other the *lower*, base.

7. A **Medial Line** of a triangle is a line drawn from the vertex of any angle to the middle point of the opposite side.

8. The **Projection** of one line upon another is the portion of the latter intercepted between perpendiculars let fall from the extremities of the former. Thus, A B and C D are the projections of E and F; and in the triangle, *a* and *b* are the projections of *c* and *d*. In the latter case, one extremity is on the line, and hence, the perpendicular from it reduces to a point.





For convenient reference, the following tables of measure are given:

I. TABLE OF LINEAR MEASURES.

12 Inches make 1 Foot.
3 Feet make 1 Yard.
5½ Yards make 1 Rod.
40 Rods make 1 Furlong.
8 Furlongs make 1 Mile.

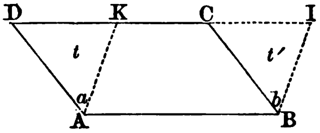
II. TABLE OF SURFACE MEASURES.

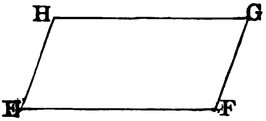
144 Square Inches make 1 Square Foot.
9 Square Feet make 1 Square Yard.
30¼ Square Yards make 1 Square Rod.
40 Square Rods make 1 Rood.
4 Roods make 1 Acre.
640 Acres make 1 Square Mile.

THEOREM I.

Parallelograms having equal bases and equal altitudes are equal.

Let the parallelograms, A B C D and E F G H, have equal bases and equal altitudes; then will they be equal.





For, let E G be applied to A C so that their lower bases shall coincide. Since they have equal altitudes their upper bases will lie in D C, or D C produced; and E G will take the position A I. In the triangles t and t',

$AK = BI$,
$AD = BC$ - - - - - - - - - *Ch. II, Th. XXIII,*
$a = b$ - - - - - - - - - - - *Ch. II, Th. XI;*
$\therefore\ t = t'$ - - - - - - - - - - - *Ch. II, Th. IV.*

Denote the whole quadrilateral, A B I D, by Q.

Then, (1) $Q = Q$.
(2) $t = t'$.
(1) − (2) = (3) $Q - t = Q - t'$.
But, $Q - t = AI$, which equals E G;
and $Q - t' = AC$.
$\therefore$ $AC = AI = EG$. *Q. E. D.*

Cor. I.—*Rectangles having equal bases and equal altitudes are equal.*

For a rectangle is a species of parallelogram, and what is true of the whole is true of a part.

Cor. II.—*A parallelogram is equal to a rectangle having an equal base and equal altitude.*

QUERIES.

1. What are the essential points in the demonstration proper?—First, to superpose the parallelograms; second, to prove t and t' equal; third, to take t and t' from the entire figure.

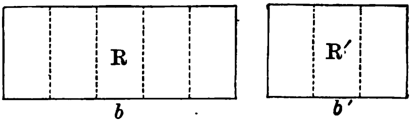
2. What principles are used in proving the equality of t and t'?

3. What axiom is involved in subtracting t and t' from the quadrilateral?

THEOREM II.

Rectangles having equal altitudes are proportional to their bases.

There may be two cases: the bases may be commensurable, or they may be incommensurable.



First: Let R and R′ be two rectangles having equal altitudes, whose bases, b and b', are commensurable; then will $R : R' :: b : b'$.

For, suppose the common unit of measure to be contained in b five times, and in b' three times. Then, evidently,

$$b : b' :: 5 : 3.$$

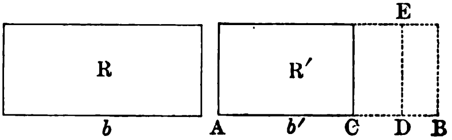
Now, conceive b to be divided into five equal parts, and b' into three, each equal to the common unit. At the points of division, erect perpendiculars to the bases. R will then be divided into five rectangles, and R′ into three, all equal, each to each, since they have equal bases and equal altitudes (*Th. I, Cor. I*); hence, $R : R' :: 5 : 3.$

But, $b : b' :: 5 : 3;$

$$\therefore \quad R : R' :: b : b'.$$

Second: Let R and R' be two rectangles whose altitudes are equal, but whose bases, b and b', are incommensurable; then will

$$R : R' :: b : b'.$$



For, if the fourth term of the proportion is not b', it is either greater or less than b'. Suppose, then, that

$$R : R' :: b : AB;$$

in which AB is *greater* than b'.

Conceive b to be divided into equal parts, each less than BC; and let one of these parts be applied, as a measure, to AB, commencing at A. Between B and C there will be at least one point of division, as D. At D, erect the perpendicular DE, meeting the upper base produced, in E. Since b and AD are commensurable, we have, by the preceding case,

$$R : AE :: b : AD.$$

But, by hypothesis, $R : R' :: b : AB$;
hence, $R' : AE :: AB : AD$ - *Ch. III, Th. VII.*
But, $R' < AE$; $\therefore AB < AD$, which is false.

Since the conclusion is false, the supposition which led to it is false; that is, the fourth term of the proportion cannot be greater than b'. By a similar process, it may be shown that the fourth term cannot be *less* than b'; and, since it can be neither greater nor less, it must be *equal* to b'. $\therefore R : R' :: b : b'$.

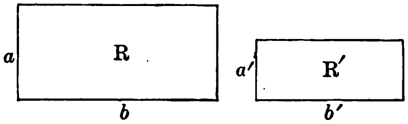
Q. E. D.

COR.—*Rectangles having equal bases, are to each other as their altitudes.*

For, the bases may be considered as the altitudes and the altitudes as the bases.

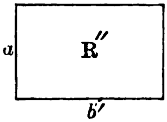
THEOREM III.

Any two rectangles are proportional to the products of their bases by their altitudes.



Let R and R' be two rectangles; then will

$$R : R' :: a\,b : a'b'.$$



For, construct a third rectangle whose altitude shall be that of R, and whose base shall be that of R'. Then, by Theorem I,

$$(1)\ R : R'' :: b : b'.$$

Since R'' and R' have equal bases, we have

$$(2)\ R'' : R' :: a : a' \quad - - - - - - \quad \textit{Th. II, Cor.}$$

$$(1) \times (2) = (3)\ R\,R'' : R'R'' :: a\,b : a'b'.$$

$$\text{But } R\,R'' : R'R'' :: R : R' \quad - - \quad \textit{Ch. III, Th. X.}$$

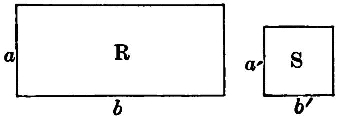
$$\therefore\ R : R' :: a\,b : a'b'. \qquad \textit{Q. E. D.}$$

QUERY.

What Problem of Ch. V is involved in the construction of R''?

THEOREM IV.

The area of a rectangle is equal to the product of its base by its altitude.



Let R be any rectangle with base b and altitude a.

Upon any assumed base, as the unit of length, construct a square, as S.

Since a square is a species of rectangle, we shall have, by Theorem III,

$$R : S :: a\,b : a'b'.$$

But a' and b' are each equal to the linear unit, and S is therefore the superficial unit, or unit of area; hence,

$$S = 1, \text{ and } a'b' = 1 \times 1 = 1.$$

$$\therefore\ 1 : S :: 1 : a'b'.$$

$$\text{but } R : S :: ab : a'b'.$$

$$\therefore\ R : 1 :: ab : 1.$$

$$\therefore\quad R = ab \text{ - - - - } \textit{Ch. III, Th. I.}$$

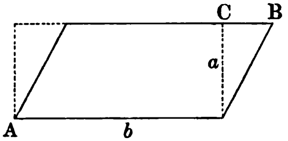
That is, the number of superficial units in the rectangle equals the number of linear units in the base multiplied by the number of linear units in the altitude. *Q. E. D.*

Cor.—*The area of a square is equal to the second power of one of its sides.*

For, in this case, the base and the altitude are equal.

THEOREM V.

The area of any parallelogram is equal to the product of its base by its altitude.



Let A B be any parallelogram, b its base, and a its altitude; then will

$$AB = a\,b.$$

For, construct the rectangle A C, having the same base and altitude as the parallelogram.

Then, $AB = AC$ - - - - - - - - - - - *Th. I, Cor. II.*

But $AC = ab$ - - - - - - - - - - - - - *Th. IV.*

$\therefore$ $AB = ab.$ *Q. E. D.*

Cor.—*Parallelograms are to each other as the products of their bases by their altitudes.*

For, let P and P′ denote any two parallelograms, b and b' their bases, a and a' their altitudes. Then, by the Theorem just proved,

$$(1)\ P = a\,b, \text{ and}$$

$$(2)\ P' = a'b'.$$

$$(1) \div (2) = (3)\ \frac{P}{P'} = \frac{a\,b}{a'b'}; \text{ whence,}$$

$$P : P' :: a\,b : a'b'.$$

EXERCISES.

1. Parallelograms having equal altitudes are proportional to their bases.

2. Parallelograms having equal bases are proportional to their altitudes.

3. A rectangular piece of land containing 6 acres is 120 rods long; required its width and the distance around it.

4. How long must a rectangular board be, which is 8 inches wide, to contain 10 square feet?

5. The area of a rhomboid is 968, and the base 120; required the altitude.

6. The sides of a rectangle are 25 and 14 feet; required to cut off an area of 154 square feet, by a line parallel to the shorter side.

7. A farmer has a field 20 rods square, from one side of which he wishes to cut a rectangular lot containing 1 acre; required the width of lot.

8. The walls of a room are 15 × 25 feet, and 10 feet high; how many square yards of plastering?

9. Can you construct quadrilaterals of different areas from sides of the same length?

10. Of four lines, two are each 6, and two each 8; required to form them into a parallelogram of largest area.

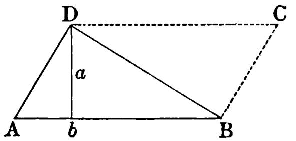
Suggestion.—The base remaining constant, when will the area be greatest?

11. Why, in the frame of a building, does a carpenter put braces in the corners? Can you alter the shape of a triangle without altering the length of at least one side?

12. The dimensions of a rectangular lot are 68 ft. 3 in. by 56 ft. 8 in.; how many stones, of rectangular form, 2 ft. 3 in. by 10 in., will be required to pave it?

THEOREM VI.

The area of a triangle is equal to one-half the product of its base by its altitude.



Let A B D be any triangle; then will

$$ABD = \tfrac{1}{2}\,ab.$$

For, draw D C parallel to A B, and B C parallel to A D, forming the parallelogram A B C D.

Then $ABCD = ab$ - - - - - - - - - - - - - *Th. V.*

But $ABD = \tfrac{1}{2}\,ABCD$ - - - - *Ch. II, Th. XXIII.*

$\therefore$ $ABD = \tfrac{1}{2}\,ab.$ *Q. E. D.*

Cor. I.—*A triangle is one-half of any parallelogram having an equal base and an equal altitude.*

Cor. II.—*Any two triangles are to each other as the products of their bases by their altitudes.*

For, let T and T′ denote two triangles whose bases are b and b', and whose altitudes are a and a'.

Then, (1) $T = \frac{1}{2}\,a\,b,$

and (2) $T' = \frac{1}{2}\,a'b'.$

$$(1) \div (2) = (3) \quad \frac{T}{T'} = \frac{\frac{1}{2}\,a\,b}{\frac{1}{2}\,a'b'} = \frac{a\,b}{a'b'}; \text{ whence,}$$

$$T : T' :: a\,b : a'b'.$$

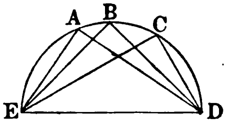
EXERCISES.

1. Triangles having equal altitudes, are proportional to their bases.

2. Triangles having equal bases, are proportional to their altitudes.

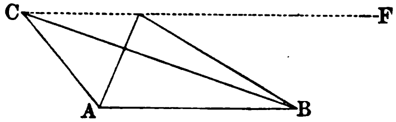
3. The area of a triangle is 60, and the base is 10; required the altitude.

4. The base of a triangular board is 9 feet, and the altitude 18 inches; required the area.



5. Which of the triangles in the annexed semicircle has the largest area?

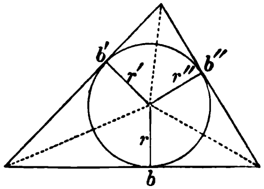
6. Can the area of a triangle be varied while the sides remain of the same length?



7. Which of the two triangles in the annexed diagram has the larger area?

8. The area of a triangle is 48: if a line be drawn from the vertex of the opposite angle to the middle point of the base, what will be the area of the partial triangles into which the given triangle is divided?

9. A framework, in the form of a triangle, has its sides united at the vertices by pivots; can the pivots play, so as to alter the form of the frame? What principle is involved?



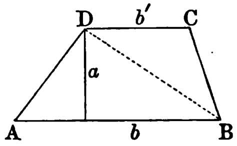
10. The gable ends of a house are forty feet wide, and the distance from the level of the eaves to the summit of the roof is 8 feet; how many square feet of lumber are required to board up the two ends?

11. The area of a triangle is equal to one-half the product of its perimeter by the radius of the inscribed circle.

12. To find the locus of the vertices of all the triangles of constant area on either side of a given base.

THEOREM VII.

The area of a trapezoid is equal to one-half the product of its altitude by the sum of its parallel sides.



Let ABCD be a trapezoid, whose altitude is a, and whose parallel sides are b and b'; then will

$$ABCD = \tfrac{1}{2} a (b + b').$$

For, draw DB, dividing the given trapezoid into two triangles, ABD and BCD.

Then, $ABCD = ABD + BCD.$

But, (1) $ABD = \tfrac{1}{2} a b,$

and (2) $BCD = \tfrac{1}{2} a b'.$

Adding, (3) $ABD + BCD = \tfrac{1}{2} a b + \tfrac{1}{2} a b',$

$\therefore\ ABCD = \tfrac{1}{2} a b + \tfrac{1}{2} a b'.$

Factoring,

$ABCD = \tfrac{1}{2} a (b + b').$ *Q. E. D.*

QUERIES.

1. How does it appear that the altitudes of ABD and BCD are equal to each other and also to that of the trapezoid?

2. What would be the position, in the diagram, of the altitude of BCD?

EXERCISES.

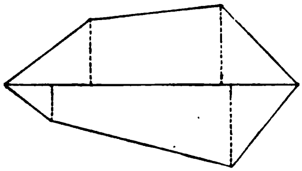
1. Required the area of a field in the form of a trapezoid, whose parallel sides are 50 and 75 rods, and the distance between them 36 rods.

2. Required the area of a board in the form of a trapezoid, whose width at one end is 2 feet 3 inches, and at the other 1 foot 6 inches, the length of the board being 16 feet.

3. The parallel sides of a trapezoid are 14 and 10, and the distance between, 6; required the area of the triangle formed by producing the inclined sides till they meet.

4. Draw a polygon, and divide it into triangles by diagonals from one vertex to the opposite vertices; then find the equation of its area by Theorem VI.

5. Divide a polygon into trapezoids and triangles, and find the equation of its area by theorems VI and VII.



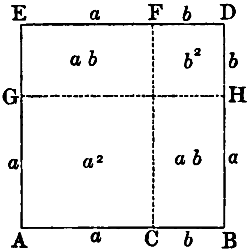
THEOREM VIII.

The square of the sum of two lines is equal to the sum of the squares of those lines plus twice their rectangle.

Let A B be the sum of two lines, A C and B C; then will

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 + 2\ AC \times BC.$$

For, on A B describe a square, A D. At C, erect the perpendicular C F. On A E, lay off A G = A C, and draw G H parallel to A B.



The whole square will then be divided into four parts,—two squares and two rectangles. Let

$a = AC$,

$b = BC$;

then $a + b = AB$.

Now, $(a+b)^2 =$ area of A D = square of A B - *Th. IV, Cor.*

$a^2 =$ area of C G = square of A C.

$b^2 =$ area of H F = square of B C.

$a\,b =$ area of G F or C H = rectangle A C × B C.

$2\,a\,b = 2$ rectangle A C × B C.

$a^2 + b^2 + 2\,a\,b =$ area of A D - - - - - - - - - *Ax. 5.*

But $(a+b)^2 =$ area of A D.

$\therefore\ (a+b)^2 = a^2 + b^2 + 2\,a\,b$

$= \overline{AC}^2 + \overline{BC}^2 + 2\,AC \times BC.$ *Q. E. D.*

Cor.—*The square of a line is four times the square of half the line.*

For, if $b = a$, the preceding equation becomes, by substitution, $(a + a)^2 = a^2 + a^2 + 2\,a \times a$

$$(2\,a)^2 = a^2 + a^2 + 2\,a^2$$

$= 4\,a^2$; in which $2\,a$ denotes the line, whence a is one-half the line.

QUERIES.

1. Why is C H a rectangle? Are its sides parallel? Are its angles right angles? By what principle? Why is B H equal to A G?

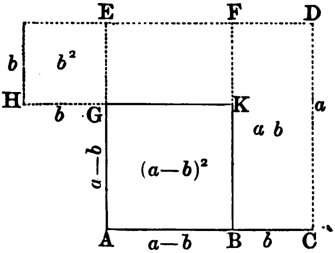
2. Why is C G a square? Are its sides parallel? This fact makes it what kind of a figure? Are its angles right angles? By what principle? *This* fact makes it what kind of a figure,—a parallelogram simply, or a rectangle? Is it also equilateral? Why?

3. E G = B C, why? Does A E = A B, and A G = A C? E F = A C, why?

4. D H = E G, why? Does D H therefore equal B C?

THEOREM IX.

The square of the difference of two lines is equal to the sum of the squares of those lines minus twice their rectangle.



Let A B be the difference of two lines, A C and B C; then will $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 - 2\,AC \times BC$.

For, on A C describe a square, A D. At B, erect the perpendicular B F. On A E, lay off A G = A B. Through G, draw H K equal and parallel to A C. Complete the square H E. Let

$a =$ A C,

$b =$ B C;

then $a - b =$ A B.

$a^2 =$ area of A D = square of A C.

$b^2 =$ area of H E = square of B C.

$(a-b)^2 =$ area of A K = square of A B.

$a\,b =$ area of H F or C F = rectangle A C × B C.

Now, the entire figure $= \text{A D} + \text{H E}$

$$= a^2 + b^2.$$

If from the entire figure, H F and C F be removed, there will remain A K. Hence,

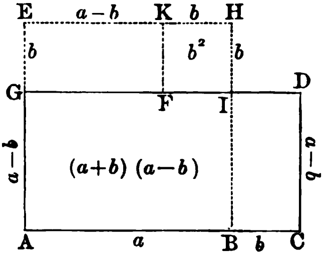
$$\begin{aligned} \text{A K} &= \text{entire figure} - (\text{H F} + \text{C F}) \\ &= a^2 + b^2 - (a\,b + a\,b) \\ &= a^2 + b^2 - 2\,a\,b. \\ \text{But A K} &= (a-b)^2 \\ \therefore \quad (a-b)^2 &= a^2 + b^2 - 2\,a\,b \\ &= \overline{\text{A C}}^2 + \overline{\text{B C}}^2 - 2\,\text{A C} \times \text{B C}. \end{aligned}$$

THEOREM X.

The rectangle of the sum and difference of two lines is equal to the difference of the squares of those lines.

Let A B and B C be two lines; then will (A B + B C) (A B − B C) $= \overline{\text{A B}}^2 - \overline{\text{B C}}^2$.

For, on A B construct a square, A H. On A E, lay off A G = A B − B C, the difference of the two lines.



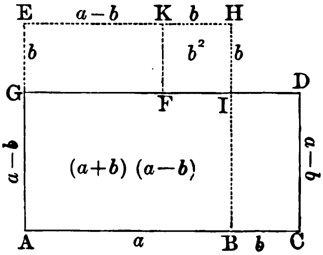
Through G, draw G D equal and parallel to A C, the sum of the two lines; and complete the rectangle A D.

Lay off H K = B C, and complete the square H F. Let

$$\begin{aligned} a &= \text{A B}, \\ \text{and} \qquad b &= \text{B C}; \\ \text{then} \qquad a + b &= \text{A C}, \\ \text{and} \qquad a - b &= \text{A G}; \\ a^2 &= \text{area of A H} = \text{square of A B}. \\ b^2 &= \text{area of F H} = \text{square of B C}. \\ (a + b)\,(a - b) &= \text{area of A D} = \text{rectangle A C} \times \text{D C}. \end{aligned}$$

Now, if F H be removed from the entire square A H, there will remain the two rectangles A I and E F; that is,

$$\text{A H} - \text{F H} = \text{A I} + \text{E F}.$$



But $EF = CI$, since they have equal bases and equal altitudes; hence,

$$AI + EF = AI + CI = AD;$$

$$\therefore \quad AH - FH = AD.$$

But, $\quad AH = a^2,$

$$FH = b^2,$$

$$AH - FH = a^2 - b^2;$$

$$\therefore \quad AD = a^2 - b^2,$$

and $AD = (a + b)(a - b)$;

$$\therefore \quad a^2 - b^2 = (a + b)(a - b);$$

that is, $\quad (a + b)(a - b) = a^2 - b^2,$

hence, $(AB + BC)(AB - BC) = \overline{AB}^2 - \overline{BC}^2$. *Q. E. D.*

Scholium.—The three preceding theorems are but geometrical conceptions and demonstrations of the algebraic *formulæ*, $(a + b)^2 = a^2 + 2ab + b^2$, $(a - b)^2 = a^2 - 2ab + b^2$, and $(a + b)(a - b) = a^2 - b^2$. The student will thence suspect the existence of an intimate relation between Algebra and Geometry.

EXERCISES.

1. The square of the sum of two lines is 49, and one of the lines is 3; required the other.
2. The square constructed upon the difference of two lines is 36, and one of the lines is 15; required the other.
3. The difference of the squares of two lines is 81, and one of the lines is 12; required the other.

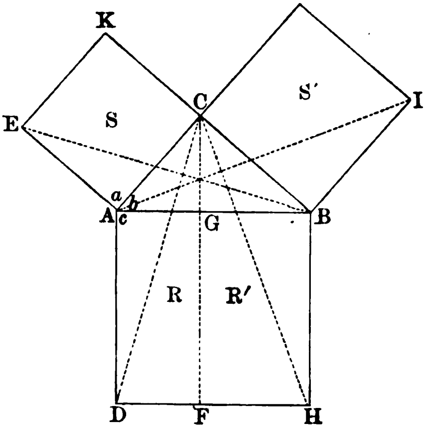
THEOREM XI.

The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

Let ABC be a right triangle, whose hypotenuse is AB; then will AH, the square of AB, be equal to S, the square of AC, plus S′, the square of BC.

For, draw CF perpendicular to DH. Then AF is a rectangle, as also BF. (*Ch. II, Th. X, Cor. II.*) Draw BE and CD.

In the triangles A B E and A C D, A E = A C, because they are sides of the same square; A B = A D, for a similar reason; $a = c$, since each is a right angle; and $b = b$; hence, $a + b = b + c$; but $a + b$ = E A B, and $b + c$ = C A D; hence E A B = C A D.



Hence the triangles are equal, since they have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each.

$$\therefore \quad (1) \quad ABE = ADC.$$

Again, the triangle A B E and the square A K have the same base, A E; and, because C K and C B form one and the same straight line (*Ch. II, Th. XV*), they have equal altitudes,—A C, and the perpendicular from B to A E produced; hence,

$$(2) \quad ABE = \tfrac{1}{2} S \quad - \; - \; - \quad \textit{Th. VI, Cor. I.}$$

The triangle A D C and the rectangle R have the same base A D, and equal altitudes,—A G, and the perpendicular from C to A D produced; hence,

$$(3) \quad ADC = \tfrac{1}{2} R.$$

But $\quad ADC = ABE;$

$\therefore \quad \tfrac{1}{2} R = \tfrac{1}{2} S;$

$\therefore \quad (4) \quad R = S.$

In a manner altogether similar, it may be shown that

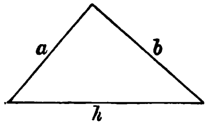
$$(5) \quad R' = S'.$$

$$(4) + (5) = (6) \quad R + R' = S + S'.$$

But $\quad R + R' =$ the square of A B, or $\overline{AB}^2$.

$\therefore \quad \overline{AB}^2 = S + S'$

$\qquad = \overline{AC}^2 + \overline{BC}^2.$ *Q. E. D.*



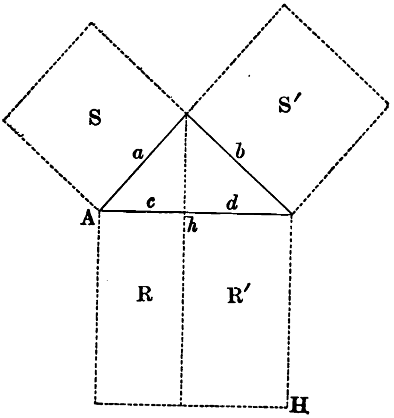
Cor. I.—*The square of either side about the right angle is equal to the square of the hypotenuse minus the square of the other side.*

For, as just proved, $h^2 = a^2 + b^2$; whence,

$$\left\{\begin{array}{l} h^2 - b^2 = a^2, \text{ and} \\ h^2 - a^2 = b^2; \text{ or,} \end{array}\right.$$

$$\left\{\begin{array}{l} a^2 = h^2 - b^2, \text{ and} \\ b^2 = h^2 - a^2 \end{array}\right.$$

Cor. II.—*The square of the hypotenuse is to the square of either side about the right angle as the hypotenuse is to the projection of that side on the hypotenuse.*



For, $AH : R :: h : c$ - - - - - - - - - *Th. II.*

But $AH = h^2$, and $R = S = a^2$;

hence, (1) $h^2 : a^2 :: h : c$.

Again, $AH : R' :: h : d$ - - - - - - - - - - *Th. II.*

But $AH = h^2$, and $R' = S' = b^2$;

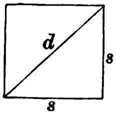
hence, (2) $h^2 : b^2 :: h : d$.

Cor. III.—*The squares of the sides about the right angle are to each other as their projections on the hypotenuse.*

For, from proportions (1) and (2), since their antecedents are in proportion, we get,

$$a^2 : b^2 :: c : d.$$

Cor. IV.—*The square of the diagonal of a square is twice the given square.*



For, the diagonal becomes the hypotenuse of a right triangle. Hence,

$$d^2 = s^2 + s^2 = 2\,s^2.$$

Cor. V.—*The diagonal of a square is to the side as the square root of 2 is to 1.*

For, the equation, $d^2 = 2\,s^2$, may be expressed in the form,

$$d^2 \times 1 = 2 \times s^2;$$

whence, $d^2 : s^2 :: 2 : 1$ - - - *Ch. III, Th. II.*

Extracting the square root of each term,

$d : s :: \sqrt{2} : 1$ *Ch. III, Th. XIV, Ex. 1.*

Cor. VI.—*The diagonal and the side of a square are incommensurable.*

This appears in the preceding proportions.

Scholium.—The discovery of this celebrated theorem is ascribed to Pythagoras (born about 600 B.C.). It is said that, as a thank-offering for its discovery, he sacrified a hundred oxen to the gods. It is of great value in the exact sciences, especially in mensuration and astronomy, where many otherwise intricate calculations are, by its aid, easily made.

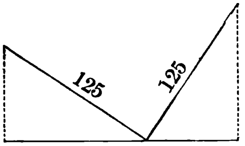
EXERCISES.

1. In a right triangle, the two sides about the right angle are 3 and 4; what is the hypotenuse?

2. A lot is in the form of a right triangle; the hypotenuse is 15, and one side 12; find the remaining side, and area.

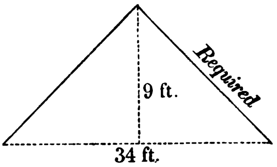
Suggestion.—If either side about the right angle be assumed as base, what is the altitude of the triangle? (*See Th. XI, Cor. I.*)

3. A ladder 125 feet long, with its foot in the street, reaches, on one side, to a window 100 feet high; on the other, to a window 75 feet high; required the width of the street.



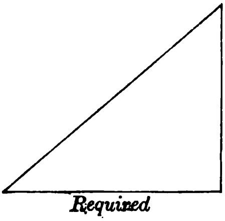
4. It is required to find the length of a rafter for a house 34 feet wide,—the ridge of the roof to be 9 feet high, above the level of the wall-plates.

Suggestion.—The form usually given to the roofs of buildings is that of the isosceles triangle.



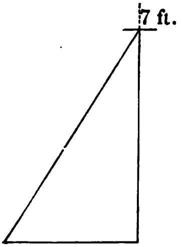
5. It is required to ascertain the greatest perpendicular height that a

roof of a given span (20 ft.) may have, when timbers of a given length (13 ft.) are to be used as rafters.



6. In a story of 8 feet, from floor to floor, a step-ladder is required, the 'strings' of which are to be of 12 feet plank; required the 'greatest run' such a length of string will afford.

7. A gate, 10 feet long and 4 feet high, requires a diagonal brace to hold it in the form of a rectangle; required the length of the brace.



8. A ladder whose length is 91 feet, stands close against a vertical wall; how far must it be drawn out at the bottom, that the top may be lowered just 7 feet?

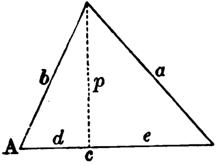
9. Two ships sail from the same port, one due west 50 miles, and the other due south 120 miles; how far are they apart?

10. Two parallel chords of a circle are 8 feet each, and their distance apart, 6 feet; required the radius of the circle. (*See Ch. IV, Ths. II and III.*)

11. Can a right triangle be constructed, whose sides shall be 3, 4, and 5? Suppose the sides were 6, 9, and 12?

THEOREM XII.

In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides minus twice the rectangle contained by one of these sides and the projection on it of the other.



Let a, b, c, be the sides of the triangle, A an acute angle, and d and e the projections of b and a on c; then will

$$a^2 = b^2 + c^2 - 2\,c\,d.$$

For, draw the perpendicular, p; then will

(1) $a^2 = p^2 + e^2$ - - - - - - - - - - - - *Th. XI.*

But $p^2 = b^2 - d^2$ - - - - - - - - - *Th. XI, Cor. I.*

And, since $e = c - d$,

$e^2 = c^2 + d^2 - 2\,c\,d$ - - - - - - - - - *Th. IX.*

Substituting in (1),

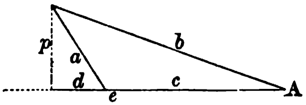
$$a^2 = b^2 - d^2 + c^2 + d^2 - 2\,c\,d$$

$$= b^2 + c^2 - 2\,c\,d.$$

Q. E. D.

EXERCISES.

1. Prove the preceding Theorem from the annexed diagram, A being the acute angle, and a the opposite side.

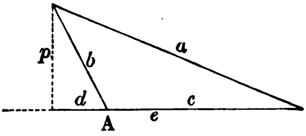


2. Two sides of a triangle are 6 and 8, and the perpendicular from the vertex of their included angle to the opposite side is 4; required the third side.

THEOREM XIII.

In any triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides plus twice the rectangle contained by one of these sides and the projection on it of the other.

Let a, b, c, be the sides of the triangle, A the obtuse angle, e and d the projections of a and b on c produced; then will



$$a^2 = b^2 + c^2 + 2\,c\,d.$$

For, draw the perpendicular, p; then will

$$a^2 = p^2 + e^2 \quad \text{- - - - - - - - - - - - -} \quad \textit{Th. XI.}$$

But $p^2 = b^2 - d^2$;

and, since $e = c + d$,

$$e^2 = c^2 + d^2 + 2\,c\,d \quad \text{- - - - - - - -} \quad \textit{Th. VIII.}$$

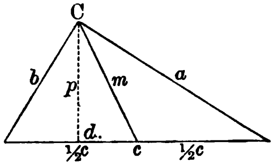
$$\therefore\ a^2 = b^2 - d^2 + c^2 + d^2 + 2\,c\,d$$

$$= b^2 + c^2 + 2\,c\,d. \qquad \textit{Q. E. D.}$$

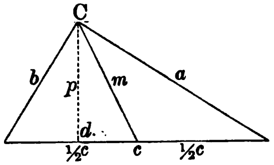
THEOREM XIV.

In any triangle, the sum of the squares of any two sides is equal to twice the square of half the third side, plus twice the square of the medial line to that side.

Let a, b, c, be the sides of a triangle, and m the medial line from C to c; then will



$$a^2 + b^2 = 2\left(\tfrac{1}{2}c\right)^2 + 2\,m^2.$$



For draw the perpendicular, p; then will

$$(1)\quad a^2 = m^2 + (\tfrac{1}{2}c)^2 + c\,d$$

And
$$(2)\quad b^2 = m^2 + (\tfrac{1}{2}c)^2 - c\,d$$

$$(1) + (2) = (3)\quad a^2 + b^2 = m^2 + (\tfrac{1}{2}c)^2 + c\,d + m^2 + (\tfrac{1}{2}c)^2 - c\,d = 2\,(\tfrac{1}{2}c)^2 + 2\,m^2.$$

Q. E. D.

QUERIES.

1 How is XIII applied? Which is the obtuse angle? The three sides?

2. How is XII applied? Which is the assumed acute angle? The three sides?

3. What is the projection of the medial line on c?

4. From what is $c\,d$, in (1) and (2) derived?

EXERCISES.

1. Subtract (2) from (1) and state, in general terms, the principle embodied in the resulting formula.

2. The base of a triangle is 50, the other sides 48 and 36; required the length of the medial line to the base.

3. In every parallelogram, the sum of the squares of the four sides is equal to the sum of the squares of the diagonals.

Suggestion.—Draw diagram, and apply preceding theorem.

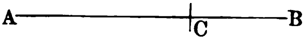
4. Two adjacent sides of a parallelogram are 49 and 56, and one of the diagonals, 64; required the other diagonal.

CHAPTER VII.

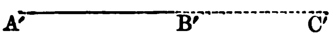
PROPORTIONALITY OF LINES.

DEFINITIONS.

1. A line is divided **internally**, when the point of division is within its extremities, as A B.



2. A line is divided **externally**, when the point of division is without its extremities, as A′ B′.

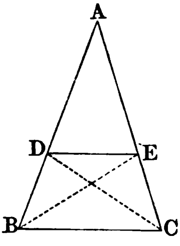


In the first case, the given line equals the sum of its segments; in the second, it equals the difference; or,

$$AB = BC + BC;\ A'B' = A'C' - B'C'.$$

THEOREM I.

A line drawn parallel to the base of a triangle divides the two sides proportionally.



Let A B C be a triangle, and D E a line parallel to the base; then will

$$AD : DB :: AE : EC.$$

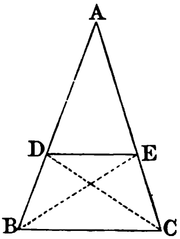
For, draw B E and C D; then the triangles A E D and D E B, having a common vertex, E, and their bases in the same straight line, A B, have a common altitude,—the perpendicular distance from E to A B; hence,

(1) A E D : D E B :: A D : D B - *Ch. VI, Th. VI, Ex. 2.*

The triangles A E D and D E C, having the common vertex, D, and their bases in A C, have also a common altitude,—the perpendicular distance from D to A C; hence,

(2) A E D : D E C :: A E : E C - *Ch. VI, Th. VI, Ex. 2.*

But the triangles, D E B and D E C, are equal, since they



have a common base, D E, and equal altitudes, —the perpendicular distances between the two parallels D E and B C (*Ch. VI, Th. VI*); hence, from (1) and (2),

∴ (3) A D : D B :: A E : E C (*Ch. III, Th. VII*).

Q. E. D.

Cor. I.—*The two sides of a triangle which are cut by a line parallel to the base, are proportional to the corresponding segments.*

For, from (3) we have

(4) A D + D B : A D :: A E + E C : A E,

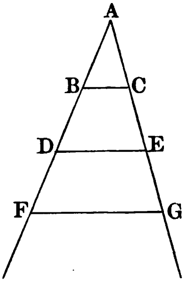
and (5) A D + D B : D B :: A E + E C : E C.

But $AD + DB = AB$, and $AE + EC = AC$;

∴ A B : A D :: A C : A E,

and A B : D B :: A C : E C.

Cor. II.—*Two lines intersected by any number of parallels, are divided proportionally.*



First: Assume the lines to intersect at A. In the triangles A D E and A F G, we have, by the theorem,

(1) A B : B D :: A C : C E;

also, (2) A D : D F :: A E : E G.

and (3) A D : B D :: A E : C E;

Since the antecedents in (2) and (3) are the same, the consequents are proportional; hence,

(4) B D : D F :: C E : E G.

From (1), A B : A C :: B D : C E;

from (4), B D : C E :: D F : E G;

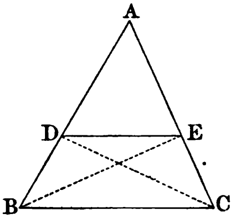
∴ (5) A B : A C :: B D : C E :: D F : E G.

Second: If the two lines are parallel, they will not intersect; but, since parallels included between parallels are equal,

BD = CE, DF = EG, whence AB = AC;
hence, the above proportion is still true.

THEOREM II.

If a straight line divides two sides of a triangle proportionally, it will be parallel to the third side.



In the triangle ABC, let DE divide AB and AC so that

AD : DB :: AE : EC;

then will DE be parallel to BC.

For, draw BE and DC. The triangles AED and BED, having the common vertex E, and their bases in the same straight line AB, have a common altitude,—the perpendicular from E to AB; hence,

AED : BED :: AD : DB - *Ch. VI, Th. VI, Ex. 2.*

ADE and CDE, having the common vertex D, and their bases in AC, have a common altitude,—the perpendicular from D to AC; hence, ADE : CDE :: AE : EC. But the second couplets of these proportions are equal, since, by hypothesis

AD : DB :: AE : EC;

∴ AED : BED :: ADE : CDE - - *Ch. III, Th. VII.*

But the antecedents of this proportion are identically the same; hence,

BED = CDE;

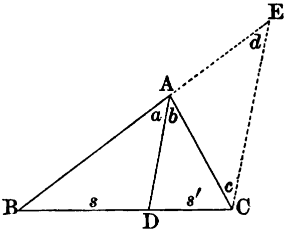
and since these triangles have a common base, DE, their altitudes are equal, that is, the perpendiculars from B and C to DE produced, are equal; hence, DE and BC are parallel.

Q. E. D.

Cor.—*If a straight line divides two sides of a triangle so that the two sides are proportional to the segments it will be parallel to the third side.*

THEOREM III.

In any triangle, the bisector of an angle divides the opposite side, internally, into segments proportional to the adjacent sides.



In the triangle A B C, let A D be the bisector of A; then will

$$s : s' :: AB : AC.$$

For, draw C E parallel to A D, and produce it until it meets B A produced, at E.

Because A D and C E are parallel, and cut by A C, $b = c$; because the two parallels are cut by B E, $a = d$; but $a = b$, by hypothesis; hence, $c = d$; hence,

$AC = AE$ - - - - - *Ch. II, Th. XVIII.*

Since, by construction, A D is parallel to C E,

$$s : s' :: AB : AE;$$

$$\text{but } AE = AC;$$

$$\therefore\ s : s' :: AB : AC.$$

Q. E. D.

Note.—This Theorem finds an important application in Physics, in determining the relation between the conjugate foci of a concave spherical mirror.

EXERCISES.

1. The sides of a triangle are 60, 50 and 40; required the segments into which the bisector of the opposite angle divides the base.

Suggestion.—Assume any side as base. From the preceding,

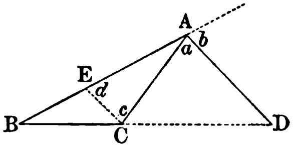
$$s + s' : s :: AB + AC : AB.$$

$$s + s' = \text{what?}$$

2. In the preceding example, find the length of the bisector. (*See Ch. VI, Ths. XII and XIII.*)

THEOREM IV.

In any triangle, the bisector of an exterior angle divides the opposite side, externally, into segments proportional to the adjacent sides.



In the triangle A B C, let A D bisect the exterior angle at A; then will

B D : C D :: A B : A C.

For, draw C E parallel to A D.

$$a = c,$$
$$b = d;$$
but $$a = b;$$
$$\therefore \quad c = d;$$
$$\therefore \quad AC = AE$$

Since C E is parallel to A D,

B D : C D :: A B : A E - - *Th. I, Cor. I.*

But A E = A C;

∴ B D : C D :: A B : A C. *Q. E. D.*

CHAPTER VIII.

SIMILARITY OF POLYGONS.

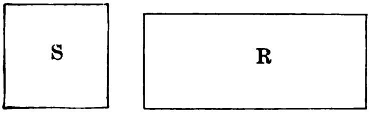
DEFINITIONS.

1. **Similar Polygons** are such as have the same form. Or, scientifically defined:

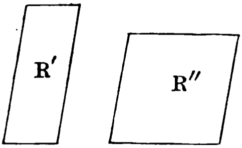
Similar Polygons are those which are mutually equiangular, and whose corresponding sides are proportional.

2. The points, lines, or angles similarly situated, in similar polygons, are said to be **homologous.**

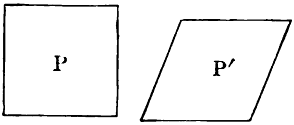
QUERIES.



1. Are S and R equiangular? Are their corresponding sides proportional? Are they, then, similar?



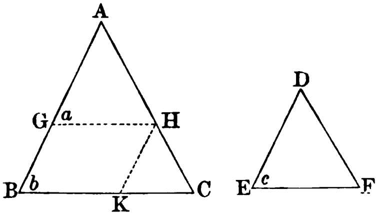
2. Are R′ and R″ equiangular? Are they similar?



3. Are P and P′ similar?

THEOREM I.

Triangles which are mutually equiangular, are similar.



In the triangles A B C and D E F, let A = D, B = E, and C = F; then will they be similar.

For, take A G = D E, A H = D F, join G H, and draw

H K, parallel to A B. The triangles A G H and D E F are equal throughout (*Ch. II, Th. IV*); hence, $a = c$, and $GH = EF$; but $b = c$; hence, $a = b$; hence, G H and B C are parallel; hence,

$AB : AG :: AC : AH$ - *Ch. VII, Th. I, Cor. I.*

But $AG = DE$,

and $AH = DF$;

$\therefore$ (1) $AB : DE :: AC : DF$.

Since H K is parallel to A B; then,

$AC : AH :: BC : BK$ - - *Ch. VII, Th. I, Cor. I.*

But $BK = GH$ (*Ch. II, Th. XXIII, Cor.*), which, as we have seen, equals E F; hence, $BK = EF$. But A H is equal to D F;

(2) $AC : DF :: BC : EF$.

Combining (1) and (2),

$AB : DE :: AC : DF :: BC : EF$.

Therefore, the triangles, being mutually equiangular, and having their corresponding sides proportional, are similar.

Q. E. D.

Cor. I.—*Two triangles are similar, when they have two angles of the one equal to two angles of the other, each to each.*

Cor. II.—*In similar triangles, the corresponding or homologous sides lie opposite equal angles.*

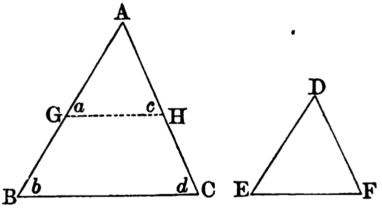
This is evident from a simple inspection of the diagrams,—noting the relative position of the proportional sides.

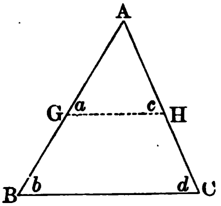
THEOREM II.

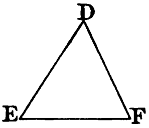
Triangles which have their homologous sides proportional, are similar.

In the triangles A B C and D E F, let $AB : DE :: AC : DF :: BC : EF$; then will they be similar.

For, take $AG = DE$, and







$AH = DF$, and draw GH. By hypothesis,

$$AB : DE :: AC : DF,$$

$$DE = AG,$$

and $$DF = AH;$$

$$\therefore\ AB : AG :: AC : AH.$$

Hence, GH is parallel to BC (*Ch. VII, Th. II, Cor.*);
hence, $a = b$, and $c = d$;
hence, AGH and ABC are equiangular;
hence, $AC : AH :: BC : GH$. But, by hypothesis,

$$AC : DF :: BC : EF.$$

But $AC : AH = AC : DF$;

$\therefore$ $BC : GH :: BC : EF$ - - - - - *Ch. III, Th. VII.*

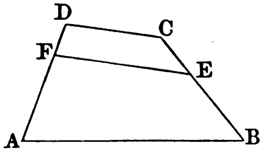
But $BC = BC$;

$\therefore$ $GH = EF$.

Hence, the triangles AGH and DEF are equiangular (*Ch. II, Th. XVII*).
But AGH is equiangular to ABC; hence, DEF is equiangular to ABC.

Therefore, DEF and ABC are similar (*Th. I*). *Q. E. D.*

Scholium.—It is thus seen that if triangles are mutually equiangular, they have their homologous sides proportional; and, conversely, if they have their homologous sides proportional, they are mutually equiangular. Either of these conditions involves the other. Hence, in establishing the similarity of two triangles, it is sufficient to prove, *either* that they are mutually equiangular, *or* that their homologous sides are proportional. To establish the similarity of other polygons, however, it will be necessary to prove *both* these conditions, since the equality of their angles does not involve the proportionality of their sides, nor conversely.



For example, EF being parallel to CD, the quadrilaterals

ABCD and ECDF are mutually equiangular, but the sides about these equal angles are not proportional; for if

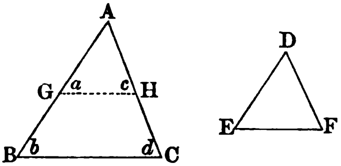
$$AD : DF :: CD : EF,$$

then, since $CD < EF$, AD would be less than DF, whereas it is greater.

THEOREM III.

Triangles which have an angle in each equal, and the including sides proportional, are similar.

In the triangles ABC and DEF, let A = D, and suppose that AB : DE :: AC : DF; then will they be similar.

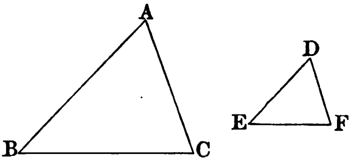


For, take AG = DE, and AH = DF; draw GH, then, since AB : DE :: AC : DF, it follows that AB : AG :: AC : AH; hence, GH is parallel to BC; hence, $a = b$, and $c = d$; hence, AGH is equiangular to ABC; but DEF is equiangular to AGH, since A = D, AG = DE, and AH = DF; hence, DEF is equiangular to ABC; hence, it is also similar to ABC. *Q. E. D.*

THEOREM IV.

Triangles which have their sides parallel, each to each, are similar.

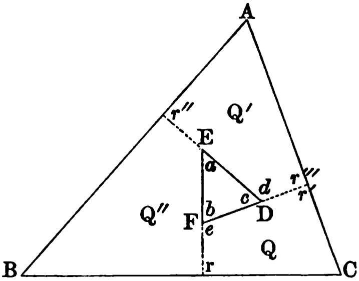
In the triangles ABC and DEF, let AB be parallel to DE, AC to DF, and BC to EF; then will they be similar.



For, since AB is parallel to DE, and BC to EF, B = E (*Ch. II, Th. XI*); since BC is parallel to EF, and AC to DF, C = F; hence, A = D; hence, the two triangles are equiangular; hence, they are similar. *Q. E. D.*

THEOREM V.

Triangles which have their sides perpendicular, each to each, are similar.



In the triangles A B C and D E F, let D E be perpendicular to A B, D F to A C, and E F to B C; then will they be similar.

For, produce the sides of D E F till they meet the sides of A B C, forming the quadrilaterals Q, Q′, Q″. The four angles of Q are together equal to four right angles (*Ch. II, Th. XXVI, Cor.*); but

$$r + r' = \text{two right angles};$$

$$\therefore\quad e + C = \text{two right angles};$$

but

$$b + e = \text{two right angles}\ (\textit{Ch. II, Th. I});$$

$$\therefore\quad b + e = e + C;$$

$$\therefore\quad b = C.$$

Again, $r'' + r''' + d + A = 4\ R$; but

$$r'' + r''' = 2\ R;\ \text{hence,}$$

$$d + A = 2\ R;\ \text{but}$$

$$c + d = 2\ R;\ \text{hence,}$$

$$c + d = d + A;\ \text{hence,}$$

$$c = A.$$

Therefore, two angles of the one triangle being equal to two angles of the other, each to each, the third angles are equal, and $a = B$; hence A B C and D E F are equiangular; hence, they are also similar. *Q. E. D.*

EXERCISES.

1. The sides of a triangle are 10, 15, and 20; the homologous sides of a second triangle are 5, 7, and 9; are they similar?

2. The sides of a triangle are 5, 7, and 9; the side of a similar triangle, homologous with 7, is 3½: required the remaining sides of the latter.

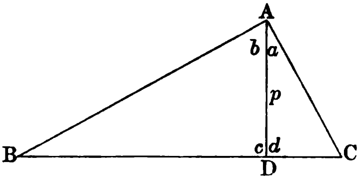
THEOREM VI.

If, in any right triangle, a perpendicular be drawn from the vertex of the right angle to the hypotenuse:

1. *The two triangles on either side of the perpendicular will be similar to the given triangle, and to each other;*

2. *Either side about the right angle will be a mean proportional between the hypotenuse and the adjacent segment;*

3. *The perpendicular will be a mean proportional between the two segments of the hypotenuse.*



Let A B C be a right triangle, and A D a perpendicular from the vertex A, of the right angle, to the hypotenuse; then

First: A B D and A D C will be similar to A B C, and to each other.

For, in the triangles A B D and A B C, the angle B is common; $c = A$, since both are right angles; hence the third angles are equal, and $b = C$; hence, A B D and A B C are equiangular; hence they are also similar.

In the triangles A D C and A B C, the angle C is common; $d = A$; hence, $a = B$; hence A D C and A B C are equiangular, and therefore similar.

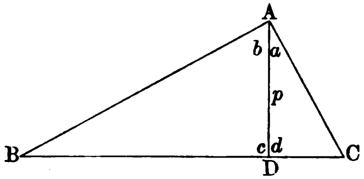
If now, A B D and A D C are each equiangular to A B C, they are equiangular to each other, and (*Th. I*) are therefore similar to each other.

Second: A B will be a mean proportional between B C and B D; and A C, between B C and D C.

For, since A B D and A B C are similar, their homologous sides are proportional; hence, B C : A B :: A B : B D.

Since A D C and A B C are similar, their homologous sides are proportional; hence, B C : A C :: A C : D C.

Third: p will be a mean proportional between B D and D C.



For, since A B D and A D C are similar, their homologous sides are proportional; hence B D : *p* :: *p* : D C; that is, B D, of the triangle A B D, is to *p*, of the triangle A D C, as *p*, of the triangle A B D, is to D C, of the triangle A D C. *Q. E. D.*

Cor. I.—*The squares of the sides about the right angle are proportional to the adjacent segments of the hypotenuse.*

For, the proportions in '*second*' give

$$(1)\quad \overline{AB}^2 = BC \times BD,$$

$$(2)\quad \overline{AC}^2 = BC \times DC.$$

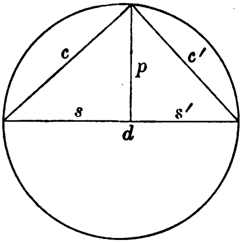
$$(1) \div (2) = (3)\quad \frac{\overline{AB}^2}{\overline{AC}^2} = \frac{BC \times BD}{BC \times DC} = \frac{BD}{DC};\ \text{hence,}$$

$$\overline{AB}^2 : \overline{AC}^2 :: BD : DC.$$

Cor. II.—*The sum of the squares of the sides about the right angle is equal to the square of the hypotenuse.*

For, by adding equations (1) and (2) above, we have

$$\overline{AB}^2 + \overline{AC}^2 = BC \times BD + BC \times DC = BC\,(BD + DC)$$
$$= BC\,(BC) = \overline{BC}^2.$$



Cor. III.—*If from any point in the circumference of a circle, a perpendicular be drawn to a diameter, and if from the same point chords be drawn to the extremities of the diameter:*

(1) *The perpendicular will be a mean proportional between the segments of the diameter;*

(2) *Either chord will be a mean proportional between the diameter and the adjacent segment;*

(3) *The sum of the squares of the chords will be equal to the square of the diameter.*

Scholium.—Corollary II shows how the Pythagorean Theorem results from the general fact of similarity.

QUERIES.

1. In the proportion B C : A B :: A B : B D, why are B C and A B homologous? To which triangle does the first A B belong?

2. In the proportion B C : A C :: A C : D C, why are B C and A C homologous?—A C and D C? To which triangle does the first A C belong?—To which the second? What equal angles lie opposite the sides A C, of the triangle A B C, and D C, of the triangle A D C?

EXERCISES.

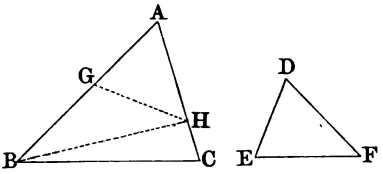
1. In a right triangle, the sides about the right angle are 12 and 15; required the length of the perpendicular from the vertex of the right angle to the hypotenuse.

Suggestion.—Find the hypotenuse and one of the segments.

2. A lot is in the form of a right triangle, the two sides about the right angle being 24 and 40; where will a perpendicular line from the vertex of the right angle cut the third side?

THEOREM VII.

Triangles which have an angle in each equal, are proportional to the rectangles of the including sides.



In the triangles A B C and D E F, let A = D; then will A B C : D E F :: A B × A C : D E × D F.

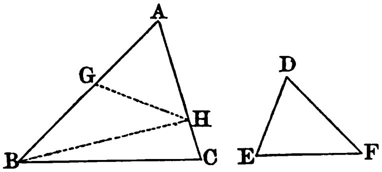
For, take A G = D E, A H = D F, and draw G H and B H. Then, the triangle A G H = D E F. Since A B H and A G H have the common vertex H, and their bases in A B,

(1) A B H : A G H :: A B : A G.

Since A B C and A B H have the common vertex B, and their bases in A C,

(2) A B C : A B H :: A C : A H.

(1) × (2) = (3) $ABC \times ABH : ABH \times AGH :: AB \times AC : AG \times AH.$



But $ABC \times ABH : ABH \times AGH = ABC : AGH$;

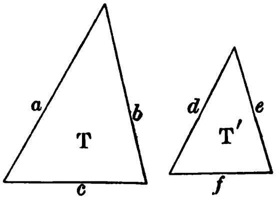
$\therefore ABC : AGH :: AB \times AC : AG \times AH.$ But $AGH = DEF$, and

$AG \times AH = DE \times DF.$

$\therefore ABC : DEF :: AB \times AC : DE \times DF.$ *Q. E. D.*

THEOREM VIII.

Similar triangles are proportional to the squares of their homologous sides.



Let T and T′ be two similar triangles; then will

$$T : T' :: b^2 : e^2.$$

For, the triangles, being similar, are equiangular; hence,

(1) $T : T' :: ab : de.$

Also, being similar, their homologous sides are proportional; hence,

(2) $a : d :: b : e$;

also, (3) $b : e :: b : e.$

(2) × (3) = (4) $ab : de :: b^2 : e^2$ - - - *Ch. III, Th. XIII.*

But $T : T' :: ab : de$;

$\therefore$ $T : T' :: b^2 : e^2$ - - - - - - *Ch. III, Th. VII.*

Q. E. D.

EXERCISES.

1. Show that $T : T' :: a^2 : d^2$.

Suggestion.—$(2)^2 =$ what?

2. Show that $T : T' :: c^2 : f^2$.

3. The areas of two triangular lots are to each other as 4 to 9; what is the ratio of their homologous sides?

Suggestion.—$T : T' :: 4 : 9$.

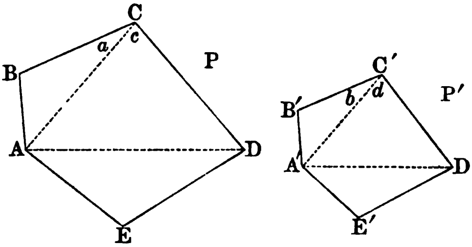
4. The base of a triangular lot is 25 rods; required the base of a similar lot of 4 times the area.

5. The homologous sides of two similar triangular fields are in the ratio of 5 to 3; how much greater is the area of the first than that of the second?

Ans. $2\frac{7}{9}$.

THEOREM IX.

Similar polygons may be resolved into the same number of triangles, similar, each to each, and similarly placed.



Let P and P′ be two similar polygons; then can they be resolved into the same number of similar triangles, placed similarly.

For, from any two corresponding vertices, as A and A′, draw diagonals to the vertices of the opposite angles.

Since the polygons are similar, they are equiangular, and their homologous sides are proportional; hence,

$$B = B', \text{ and } AB : A'B' :: BC : B'C'.$$

Hence, A B C and A′B′C′ are similar (*Th. III*); if similar, they are equiangular; hence, $a = b$; hence, $c = d$, since $a + c = b + d$, by definition.

Also, $AC : A'C' :: BC : B'C'$;

but $BC : B'C' :: CD : C'D'$;

$\therefore$ $AC : A'C' :: CD : C'D'$.

Hence, A C D and A′C′D′ are similar (*Th. III*).

In A E D and A′E′D′, $E = E'$, and

$$AE : A'E' :: ED : E'D';$$

hence, they are similar. *Q. E. D.*

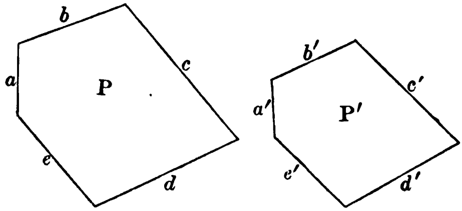
EXERCISE.

Two polygons, composed of similar triangles, similarly placed, are similar.

Suggestion.—It is to be proved that the two polygons are equiangular, and have their homologous sides proportional. What are similar triangles?

THEOREM X.

The perimeters of similar polygons are proportional to any two homologous sides.



Let P and P′ be two similar polygons; then will their perimeters be proportional to any two homologous sides, as a and a'.

For, by definition,

$$a : a' :: b : b' :: c : c' :: d : d' :: e : e'.$$

$$\therefore\quad a + b + c + d + e : a' + b' + c' + d' + e' :: a : a';$$

or, denoting the perimeters by p and p',

$$p : p' :: a : a' :: b : b', \text{ etc.}$$

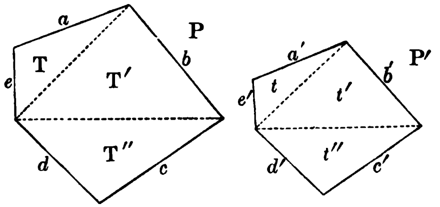
Q. E. D.

EXERCISE.

Perimeters of similar polygons are proportional to their homologous diagonals. (*See Th. IX.*)

THEOREM XI.

Similar polygons are proportional to the squares of any two homologous sides.



Let P and P′ be similar polygons; then will they be proportional to the squares of any two homologous sides, as a and a'.

For, since T and t are similar (*Th. IX*),

(1) $T : t :: a^2 : a'^2$ (*Th. VIII.*)

Since T′ and t' are similar, (2) $T' : t' :: b^2 : b'^2$.

Since T″ and $t''$are similar, (3) $T'' : t'' :: c^2 : c'^2$. But since the polygons are similar,

(4) $a : a' :: b : b' :: c : c'$.

$(4)^2 =$ (5) $a^2 : a'^2 :: b^2 : b'^2 :: c^2 : c'^2$ - *Ch. III, Th. XIV.*

∴ $T : t :: T' : t' :: T'' : t''$ - *Ch. III, Th. VII.*

∴ $T + T' + T'' : t + t' + t'' :: T : t$; that is,

$P : P' :: T : t$.

But $T : t :: a^2 : a'^2$;

∴ $P : P' :: a^2 : a'^2$. *Q. E. D.*

EXERCISES.

1. Show that $P : P' :: b^2 : b'^2 :: c^2 : c'^2 :: d^2 : d'^2 :: e^2 : e'^2$.

Suggestion.—Are the homologous sides proportional?

2. Similar polygons are proportional to the squares of their homologous diagonals.

Suggestion.—Are the diagonals sides of similar triangles?

3. If, in the proportion $P : P' :: a^2 : a'^2$, $a = a'$, what relation obtains between P and P′? State the result in general terms, as a principle.

4. If similar polygons be constructed on the three sides of a right triangle as homologous sides, the polygon on the hypotenuse will equal the sum of the polygons on the other two sides.

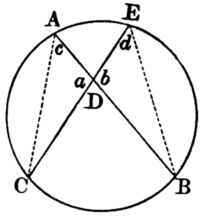
Suggestion.—If P, P′, P″, denote the polygons, h, a, and b the hypotenuse and sides, then $P : P'' :: h^2 : b^2$, and $P' : P'' :: a^2 : b^2$.

5. A field is 40 by 15 rods; what are the dimensions of a similar field, similarly situated, whose area is 9 times as great?

6. The homologous sides of two similar fields are in the ratio of 3 to 5, and the sum of their areas is 416; required their separate areas.

THEOREM XII.

If two chords intersect in a circle, the rectangle of the segments of one is equal to the rectangle of the segments of the other.



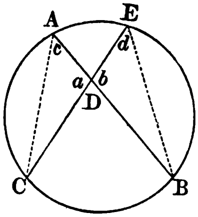
Let A B and C E be two chords intersecting in D; then will

$$AD \times DB = CD \times DE.$$

For, draw A C and B E; then in the triangles A D C and E D B,

$a = b$;

$c = d$, since each equals $\frac{1}{2}\underset{\smile}{BC}$; and, hence, the third angles are equal.



Hence, the triangles are similar.

Hence, A D : D E :: C D : D B.

Hence, A D × D B = C D × D E (*Ch. III, Th. I*). *Q. E. D.*

QUERIES.

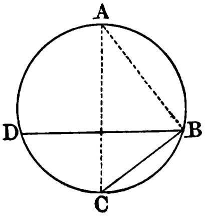
1. How do you determine what sides are homologous in A D C and E D B? What must be the relative position of such homologous sides? (*See Th. I, Cor. II.*) Why are C D and B D homologous?

2. What three leading steps or results do you discover in the demonstration proper?

3. How could you express the preceding proportion as a *principle?*

EXERCISES.

1. The two segments of a chord intersected by another chord are 7 and 9, and one segment of the latter is 3; required the length of the latter.

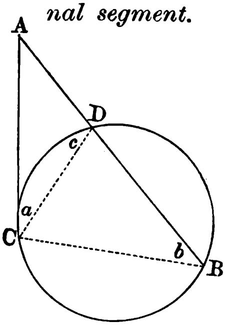


2. The chord of an arc is 80, and the chord of half the arc is 41; required the diameter of the circle.

Suggestion.—What kind of a triangle is A B C? How does A C cut D B? (*See Th. VI.*)

THEOREM XIII.

If from a point without a circle, a tangent and a secant be drawn, terminating in the circumference, the square of the tangent will equal the rectangle of the secant and its external segment.



Let A C and A B be the tangent and secant, from the point A; then will

$$\overline{AC}^2 = AB \times AD.$$

For, draw B C and C D, forming the triangles A B C and A C D.

In these triangles, angle A is common;

$a = b$, since each equals $\frac{1}{2}\,\overset{\frown}{CD}$;

hence, $c = C$.

Hence, A B C and A C D are similar.

Hence, A B : A C :: A C : A D.

∴ $\overline{AC}^2 = AB \times AD$. *Q. E. D.*

QUERIES.

1. In the triangles A B C and A C D, what side is homologous to A B? As homologous to A B, to which triangle does A C belong? What side is homologous to A D? Why?

2. How will you express the preceding proportion as a *principle?*

THEOREM XIV.

If from a point without a circle two secants be drawn terminating in the circumference, the rectangles of each secant and its external segment are equal.

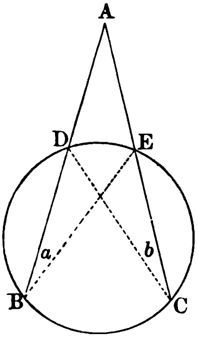
Let A B and A C be two secants from the point A, without the circle; then will

$$AB \times AD = AC \times AE.$$

For, draw B E and C D, forming the triangles A B E and A D C. In these triangles, $a = b$, since each equals $\frac{1}{2}\,\overset{\frown}{DE}$; A is common; hence, the third angles are equal, and the triangles are similar.

Hence, A B : A C :: A E : A D.

∴ $AB \times AD = AC \times AE$.



Q. E. D.

QUERIES.

1. Why are A B and A C, A E and A D, homologous sides of A B E and A D C?

2. Referring the preceding proportion to the secants, how might it be interpreted and expressed as a *principle?*

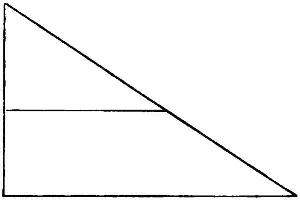
GENERAL EXERCISES.

1. From a point without a circle two secants are drawn, whose external segments are 7 and 9, the internal segment of the latter being 13; required the length of the former secant.

2. The dimensions of a rectangular field are 36 by 25, and it is desired to construct a square field of equal area; what must be the length of the side?

3. A meadow, of 1 acre, is in the form of a square; what must be the length of a rope that will permit a horse, from a point equidistant from each angle, to graze over the entire meadow.

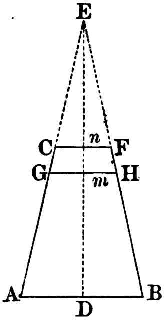
Suggestion.—At what point do the diagonals of a square cut each other? (*See Ch. VI, Th. XI, Cor. IV.*)



4. The area of a square yard is 16 square rods, and it is desired to make a walk around it that shall occupy ¼ the yard; what must be the width of the walk?

5. A right triangle has a base of 16 rods, and the perpendicular is 12 rods; and it is desired to cut off, by a line parallel to the base, a triangle whose area shall be 24 square rods: required the sides of the triangle.

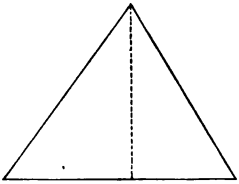
Suggestion.—Are the triangles similar? (*See Th. VIII.*)



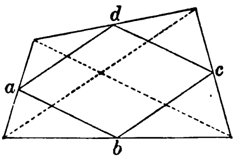
6. The equal sides of an isosceles triangle are 15 each, and the base 10; required the altitude of a right triangle constructed on the same base, and having an equal area.

7. A board is 12 feet long, 10 inches wide at the greater end and 6 inches at the less; what length must be cut off from the less end, to make a square foot?

Suggestion.—Find E *n* by similar triangles; then the area of E C F, then of E G H; then apply Th. VIII, (Ch. VIII) to find E *m*. E *m* — E *n* = required length.

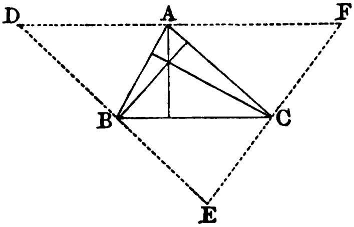


8. If a perpendicular be drawn from the vertical angle of a triangle to the base, the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.



9. The figure formed by joining the adjacent middle points of the sides of a quadrilateral is a parallelogram.

Suggestion.—Do *c d* and *a b* cut the adjacent sides proportionally? (*See Th. VIII.*)



10. The three perpendiculars from the vertices of a triangle to the opposite sides meet in the same point.

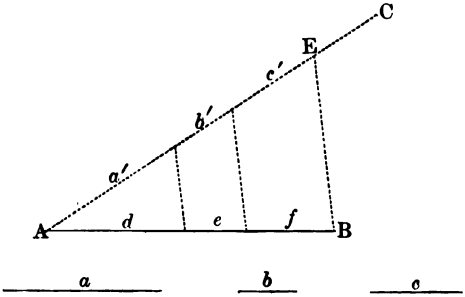
Suggestion.—Through the vertices of the triangle draw parallels to the opposite sides; and show that the perpendiculars are the loci of E, D, and F. What kind of figures are A B C F and A C B D? Is A F thence equal to A D? (*See Ch. IV, Th. XII, Ex. 8.*)

CHAPTER IX.

CONSTRUCTIONS.

PROBLEM I.

To divide a line into parts proportional to given lines.



Let A B be the given line, and a, b, c, the given lines.

Through one extremity of the given line, as A, and at any angle with it, draw an indefinite line A C.

Commencing at A, lay off on A C $a' = a$, $b' = b$, $c' = c$; and join the last point of division, E, with the second extremity of the given line, B.

Through the other points of division, draw lines parallel to E B (*Ch. V*, P*rob. VIII*).

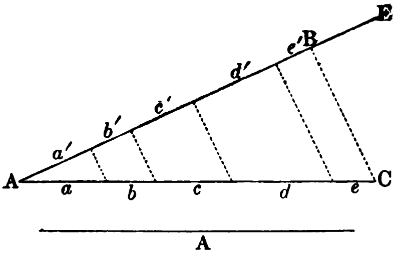
Then d, e, and f are the required parts.

For, $d : a' :: e : b' :: f : c'$; or
$d : e :: a' : b'$, and
$e : f :: b' : c'$ - - *Ch. VII, Th. I, Cor. II.*
$\therefore$ $d : e : f :: a' : b' : c' :: a : b : c$. *Q. E. F.*

PROBLEM II.

To divide a given line into the same proportional parts as a given divided line.

Let A be the given line, and A C the given divided line, whose parts are a, b, c, d, and e.



Through one extremity A, of A C, draw A E, of indefinite length, and from A lay off A B = A. Join the extremities B and C.

Through the points of division in A C, draw lines parallel to C B.

Then will a', b', c', d' and e' be the required parts.

For, $a' : a :: b' : b :: c' : c :: d' : d :: e' : e$; or

$a' : b' :: a : b$,

$b' : c' :: b : c$,

$c' : d' :: c : d$, and

$d' : e' :: d : e$. - - *Ch. VII, Th. I, Cor. II.*

$\therefore\ a' : b' : c' : d' : e' :: a : b : c : d : e$.

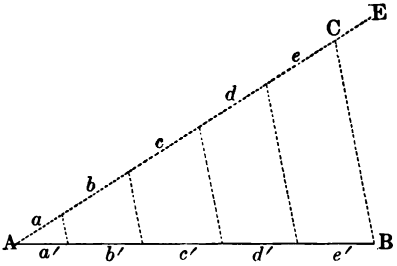
Then on the given line A, commencing at the left, lay off in succession a', b', c', d', e'. *Q. E. F.*

QUERY.

Beginning at the *right* of A, in what order should the parts be laid off ?

PROMBLEM III.

To divide a given line into any number of equal parts.



Let A B be the given line, and five the number of parts.

Through one extremity A, at any angle with A B, draw A E.

Assume any convenient unit of length, and from A, lay it off five times on A E.

Through the last point of division draw C B.

Through the other points of division draw parallels to C B.

The intercepts on A B are the required parts.

For, $a : a' :: b : b' :: c : c' :: d : d' :: e : e'$.

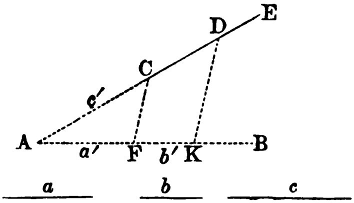
But, $a = b = c = d = e$;

$\therefore$ $a' = b' = c' = d' = e'$ - - *Ch. III, Th. IX, Ex. 1.*

Q. E. F.

PROBLEM IV.

To find a fourth proportional to three given lines.



Let *a*, *b* and *c* be the given lines.

From any point, at any angle with each other, draw two indefinite lines, A B and A E.

From A, lay off on the first, $a' = a$, $b' = b$; on the second, $c' = c$.

Join C and F.

Through the extremity of b', draw K D parallel to F C.

Then C D is the fourth proportional required.

For, $a' : b' :: c' : CD$ - - *Ch. VII, Th. I, Cor. II.*

$\therefore$ $a : b :: c : CD$. *Q. E. F.*

QUERIES.

1. Of the given lines *a*, *b*, and *c*, the *first* and *second* were laid off where? The *third ?*

2. Do we first join the extremities of the *first* and *second*, or of the *first* and *third ?*

3. The order of the proportionals being *a*, *b*, *c*, why should we expect C D to be *less* than *c ?* *(See Ch. III, Th. IX.)*

4. Does the order of the proportionals affect the result?

5. Suppose the order were, $b : a :: c :$ *the fourth line ?*

6. Suppose it were, $c : a :: b :$ *the fourth line ?*

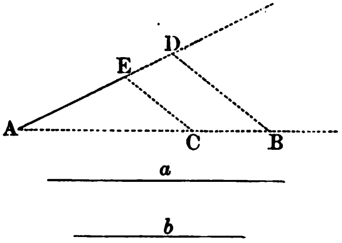
7. Could b', the less, have been laid off on a', the greater ? Would this have changed the result? Try it.

PROBLEM V.

To find a third proportional to two given lines.

Let a and b be the given lines; it is required to find a third, such that

$a : b :: b :$ *the third proportional.*



From any point, as A, at any angle, draw two indefinite lines.

Lay off $AB = a$, $AC = b$, and $AD = AC = b$.

Join B and D; and through C, draw C E parallel to B D.

Then A E is the third proportional required.

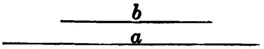
For, $AB : AC :: AD : AE$ - - *Ch. VII, Th. I, Cor. I.*

that is, $a : b :: b : AE$. *Q. E. F.*

Scholium.—This, it may be noticed, is but a particular case of Problem IV, in which the second and third of the given lines are *equal.*

QUERY.

Suppose the two lines were given as annexed, in the order $b : a :: a :$ *the third proportional?* Would the required proportional be greater or less than before?



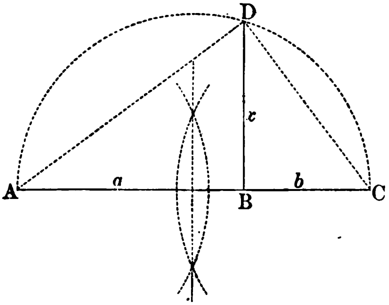
PROBLEM VI.

To find a mean proportional between two given lines.

Let a and b be the given lines.

Draw $AB = a$, and produce it till $BC = b$. On A C as a diameter, describe a semicircumference.

At B, the point of division between a and b, erect a perpendicular. (*Ch. V, Prob. I.*)



This perpendicular is the required mean proportional.

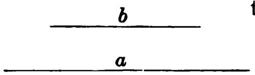
For, drawing A D and C D, A D C is a right triangle. Hence,

$a : x :: x : b$ - - *Ch. VIII, Th. VI, Cor. III.*

Q. E. F.

QUERIES.

1. The order of parts being as given, what should have led you to expect that x would be greater than the fourth term?



2. Suppose the order were as follows:

$b : x :: x : a.$

3. The two lines remaining constant, will the value of x be the same, whatever be the order of parts?

4. From the preceding proportion, what equation expresses the numerical value of x?

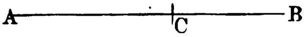
5. Regarding $a\,b$ as the product of two ***numbers***, what does B D represent?

6. Can arithmetic express exactly in numbers the square root of 15? Can geometry ***represent*** it?

7. If A B = 5, and B C = 3 (that is, if A B contains 5 units of length, and B C 3 units), how may you numerically express B D, which is then, geometrically, the square root of their product (15)?

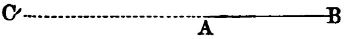
DEFINITIONS.

1. When a line is divided internally into two segments such that the greater is a mean proportional between the whole line and the less segment, it is said to be divided ***internally in extreme and mean ratio.***



Thus, A B is divided internally in extreme and mean ratio at C, if A B : A C :: A C : B C.

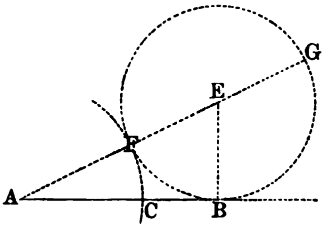
2. When a line is divided externally into two segments such that the less external segment is a mean proportional between the whole line and the greater, it is said to be divided ***externally in extreme and mean ratio.***



Thus A B is divided, externally, in extreme and mean ratio at C′, if A B : A C′ :: A C′ : B C′.

PROBLEM VII.

To divide a given line internally in extreme and mean ratio.



Let A B be the given line.

At the extremity B, erect the perpendicular $BE = \frac{1}{2} AB$.

From E as centre, with radius B E, describe a circle.

Through centre E, draw A G, cutting circumferance in F.

On A B, cut off $AC = AF$.

Then A B is divided internally in extreme and mean ratio at C.

For, A B, being perpendicular to the radius E B at its extremity, is tangent to the circle.

Hence, $AG : AB :: AB : AF$ - *Ch. VIII, Th. XIII.*

Hence, $AG - AB : AB :: AB - AF : AF$ *Ch. III, Th. VI.*

But, $AB = 2\,BE =$ diameter F G;

and, $AF = AC$.

Hence, $AG - AB = AG - FG = AF = AC$.

Hence, $AC : AB :: AB - AC : AC$.

But, $AB - AC = BC$.

Hence, $AC : AB :: BC : AC$.

$\therefore$ $AB : AC :: AC : BC$. *Q. E. F.*

QUERIES.

1. How is B E taken $= \frac{1}{2}$ A B? (*See Ch. V, Prob. IV.*)

2. What problem is involved in the construction of the perpendicular B E?

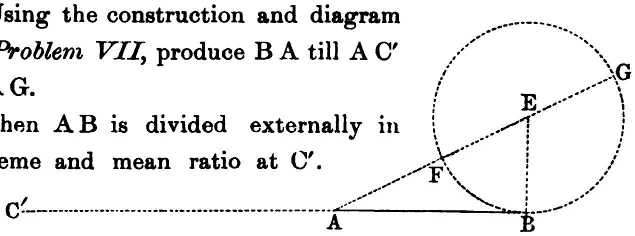
PROBLEM VIII.

To divide a given line externally in extreme and mean ratio.

Let A B be the given line.

Using the construction and diagram of *Problem VII*, produce B A till A C′ = A G.

Then A B is divided externally in extreme and mean ratio at C′.



For, A G : A B :: A B : A F; whence,

A G + A B : A G :: A B + A F : A B.

But, A G + A B = A C′ + A B = B C′; and A B + A F = F G + A F = A G = A C′.

Hence, B C′ : A C′ :: A C′ : A B; or

A B : A C′ :: A C′ : B C′. *Q. E. F.*

EXERCISES.

1. In Prob. VII, show that A C = $\frac{1}{2}$ A B ($\sqrt{5}$ − 1), and B C = $\frac{1}{2}$ A B (3 − $\sqrt{5}$).

Suggestion.—A C = A E − E F, and B E = $\frac{1}{2}$ A B. Find A E from triangle A B E.

2. In Prob. VIII, show that A C′ = $\frac{1}{2}$ A B ($\sqrt{5}$ + 1), and B C′ = $\frac{1}{2}$ A B (3 + $\sqrt{5}$).

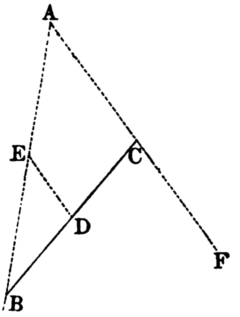
Suggestion.—A C′ = A E + E G, and B C′ = A B + A C′.

PROBLEM IX.

Through a given point in a given angle, to draw a straight line, so that the segments between the point and the sides of the angle shall be equal.

Let A be the given angle, and D the given point.

Through D, draw D E parallel to F A. Lay off E B = E A; and draw B C through the given point, which will be the required line.



For, D E being parallel to A C,

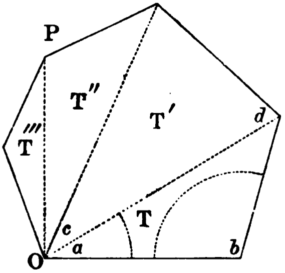
B E : E A :: B D : D C.

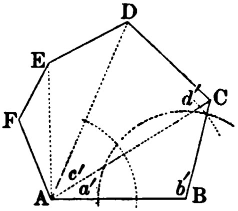
But, B E = E A;

∴ B D = D C. *Q. E. F.*

PROBLEM X.

On a given straight line, to construct a polygon similar to a given polygon.





Let P be the given polygon, and A B the given line.

Divide the given polygon into triangles, by diagonals drawn from O. At the extremities A and B, of the given line, construct $a' = a$, and $b' = b$. Produce A C and B C till they intersect. Then A B C must be similar to T. At the points A and C in the line A C, construct $c' = c$, and $d' = d$; and produce the sides till they intersect in D.

Then A D C is similar to T'. Similarly, construct A D E and A E F similar to T'' and T'''. Then the entire polygon A B C D E is similar to P - - - - - - - - - *Ch. VIII, Th. IX, Ex.*

Q. E. F.

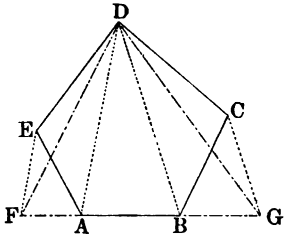
PROBLEM XI.

To construct a triangle equal to a given polygon.

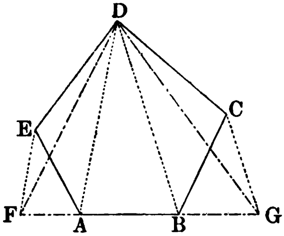
Let A B C D E be the given polygon.

Draw the diagonal D A. Draw E F parallel to D A, meeting A B produced, in F. Join D and F.

The triangles A F D and A E D have the same base A D and the same altitude,—the perpendicular distance between the two parallels A D and F E.



∴ (1) AFD = AED.
But, (2) ABCD = ABCD.
(1) + (2) = (3) AFD + ABCD = AED + ABCD.
But, AFD + ABCD = quadrilateral BCDF
and AED + ABCD = the given polygon.
Hence, BCDF = ABCDE.



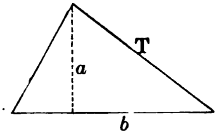
Again, draw the diagonal DB. Draw CG parallel to DB, to meet AB produced, in G. Join D and G.

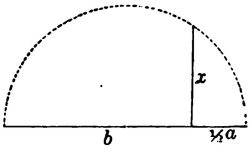
The triangles BGD and BCD have the same base BD, and their vertices in the parallel CG;

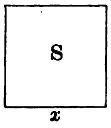
∴ (1) BGD = BCD.
But, (2) BDF = BDF.
(1) + (2) = (3) BGD + BDF = BCD + BDF.
But, BGD + BDF = triangle GDF.
and BCD + BDF = quadrilateral BCDF.
∴ GDF = BCDF.
But, BCDF = ABCDE;
∴ GDF = ABCDE. *Q. E. F.*

PROBLEM XII.

To construct a square equal to a given triangle.







Let T be the given triangle; its base b, and its altitude a. Construct a mean proportional between b and $\frac{1}{2}a$ (*Prob. VI*). On this mean proportional as a side, construct a square, which will be the square required.

For, $b : x :: x : \frac{1}{2} a$; whence

$x^2 = b \times \frac{1}{2} a$.

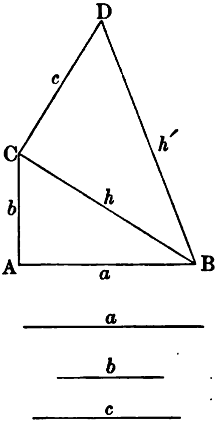
But, $x^2 =$ area of S, and

$b \times \frac{1}{2} a =$ area of T;

$\therefore$ $S = T$. *Q. E. F.*

PROBLEM XIII.

To construct a square equal to the sum of two or more given squares.



Let a, b, c, be the sides of the given squares.

Draw $AB = a$.

At one extremity, as A, erect the perpendicular $AC = b$ - - - *Ch. V, Prob. IV.*

Draw B C. Perpendicular to B C at C, draw $CD = c$; and draw B D, which will be the side of the required square.

For, $h'^2 = h^2 + c^2$ - - *Ch. VI, Th. XI.*

But $h^2 = a^2 + b^2$;

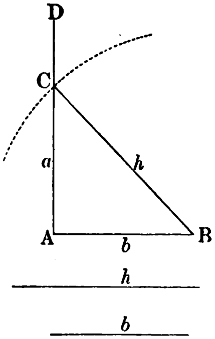
$\therefore$ $h'^2 = a^2 + b^2 + c^2$

= square on a + square on b + square on c.

Q. E. F.

PROBLEM XIV.

To construct a square equal to the difference of two given squares.



Let h and b be the sides of the given squares.

Draw $AB = b$. At one extremity, as A, erect an indefinite perpendicular, A D.

From centre B, with radius $= h$, describe an arc cutting A D in C.

Then A C is the side of the required square.

For, $a^2 = h^2 - b^2$ (*Ch. VI, Th. XI, Cor. I*);

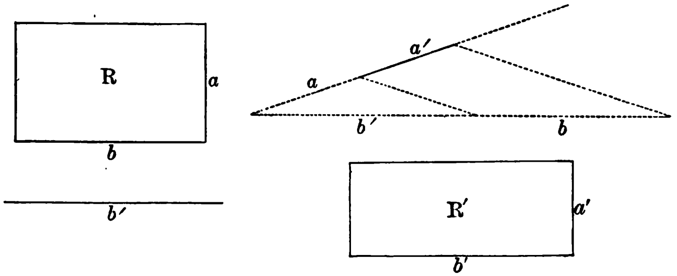
that is, square on a = square on h — square on b. *Q. E. F.*

QUERY.

How, then, may a square be constructed equal to the sum of any number of given polygons, or to the difference of any two?

PROBLEM XV.

On a given straight line, to construct a rectangle equal to a given rectangle.



Let b' be the given line, and R the given rectangle, with base b and altitude a.

Find a fourth proportional to b', b, and a - - *Prob. IV.*

With base b' and altitude a', construct R', which will be the rectangle required.

For, from the construction,

$$b' : b :: a : a'; \text{ whence,}$$

$$a'b' = a\,b.$$

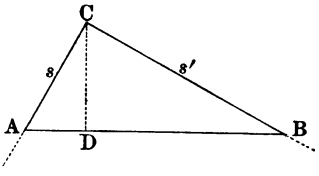
$$\text{But, } a'b' = R',$$

$$\text{and } a\,b = R;$$

$$\therefore \quad R' = R. \qquad Q.\ E.\ F.$$

PROBLEM XVI.

To find two straight lines that shall be to each other as the areas of two given polygons.



Construct two squares equal to the given polygons P and P'. Let s and s' be the sides of these squares.

Construct a right angle A C B, and lay off $AC = s$, and $BC = s'$.

Join A and B.

Drop the perpendicular C D.

A D and D B are the required lines.

For, $s^2 : s'^2 :: AD : DB$ - - - - *Ch. VI, Th. XI, Cor.*

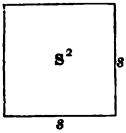
But, by construction,

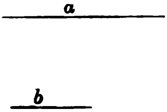
$$s^2 : s'^2 :: P : P'.$$

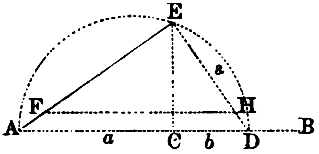
$$AD : DB :: P : P'.$$ *Q. E. F.*

PROBLEM XVII.

To find a square which shall be to a given square in the ratio of two given lines.







Let s^2 be the given square, a and b the given lines.

Draw the indefinite line, A B.

Lay off $AC = a$, and $CD = b$.

On A D as diameter, describe a semicircle.

At C, erect the perpendicular C E.

Draw E A and E D.

On E D, lay off $EH = s$; and through H, draw H F parallel to D A.

Then E F is the side of the required square.

For, $EF : EH :: EA : ED$ - - - *Ch. VII, Th. I, Cor. I.*

$\therefore \quad \overline{EF}^2 : \overline{EH}^2 :: \overline{EA}^2 : \overline{ED}^2.$

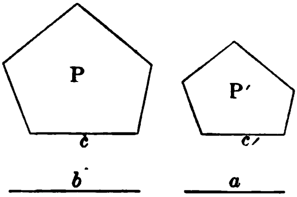
But, $\overline{EH}^2 = s^2,$

and $\overline{EA}^2 : \overline{ED}^2 :: a : b,$ - - - *Ch. VI, Th. XI, Cor. III.*

$\therefore \quad \overline{EF}^2 : s^2 :: a : b.$ *Q. E. F.*

PROBLEM XVIII.

To construct a polygon similar to a given polygon, whose area shall be in a given ratio to that of the given polygon.



Let P be the given polygon, and $a : b$ the given ratio.

Find the side c' of a square such that $c'^2 : c^2 :: a : b$. (*Prob. XVII.*)

On c', construct P′ similar to P. (*Prob. X.*)

Then, $P' : P :: c'^2 : c^2$ - - - - - - - *Ch. VIII, Th. XI.*

But, $c'^2 : c^2 :: a : b$, by construction.

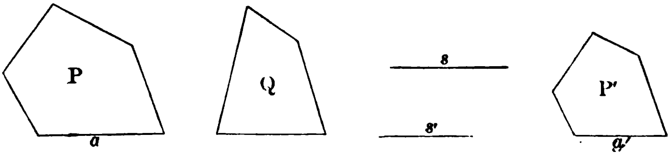
$\therefore$ $P' : P :: a : b$. *Q. E. F.*

QUERY.

In finding c', how should you know that it would be *less* than c?

PROBLEM XIX.

To construct a polygon similar to one and equal to another.



Let P be one of the given polygons, and Q the other; it is required to construct a polygon similar to P, and equal to Q.

Construct a square equal to P, and another equal to Q (*Probs. XI, XII*).

Let s be a side of the first square, and s' a side of the second.

Let a denote a side of P.

To s, s', and a, construct a fourth proportional, a'.

On a', construct a polygon, P′, similar to P.

Then P′ is the polygon required.

For, $s : s' :: a : a'$; whence,

$s^2 : s'^2 :: a^2 : a'^2$

But $s^2 = P$,

and $s'^2 = Q$;

$\therefore$ $P : Q :: s^2 : s'^2$;

$\therefore$ $P : Q :: a^2 : a'^2$.

But, $P : P' :: a^2 : a'^2$ - - *Ch. VIII, Th. XI.*

$\therefore$ $P : P' :: P : Q$.

But, $P = P$;

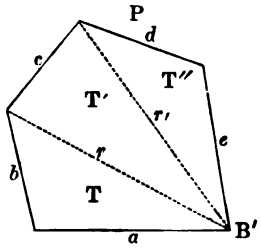
$\therefore$ $P' = Q$. *Q. E. F.*

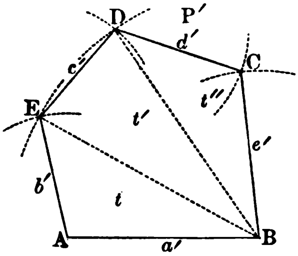
QUERY.

How should you know that the fourth proportional to s, s' and a, would be *less* than a?

PROBLEM XX.

To construct a polygon equal and similar to a given polygon.





Let P be the given polygon. From B′ draw diagonals r and r', to each of the opposite angles.

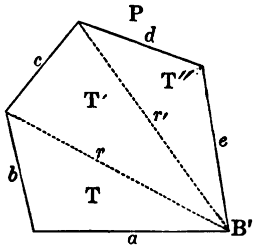
Take $a' = a$. From centre A, with radius b, describe an arc. From centre B, with radius r, describe a second arc cutting the first in E.

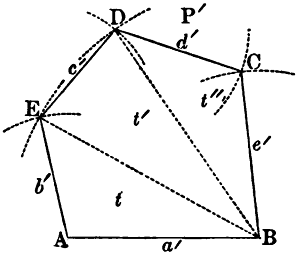
Draw A E and B E.

In like manner, on B E construct $t' = T'$; and on B D, $t'' = T''$ - - - - - - - - - - - - - - - *Ch. V, Prob. XII.*

Then P′ is the polygon required.

For, $t = T$, $t' = T'$, and $t'' = T''$ - - - - *Ch. II, Th. XX.*
Hence, P′ *equals* P.





And, since t, t' and t'' are respectively similar to T, T′ and T″, P′ is also *similar* to P - - - - - - *Ch. VIII, Th. IX, Ex.*

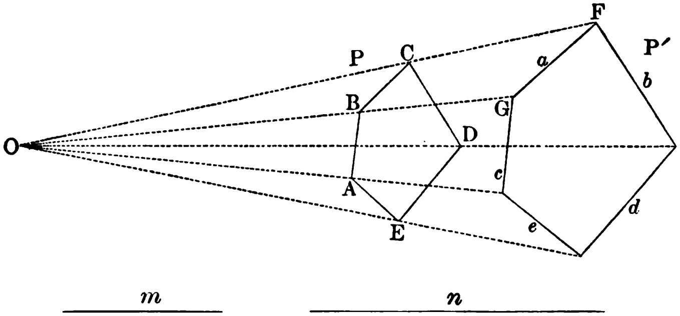
Q. E. F.

QUERY.

What is the numerical ratio of the homologous sides of t and T? Of t' and T′? Of t'' and T″?

PROBLEM XXI.

To construct a polygon similar to a given polygon, whose sides shall be to those of the given one in the ratio of two given lines.



Let P be the given polygon, and $m : n$ the given ratio.

From any point, O, draw lines through the vertices A, B, C, D, and E.

On any one of these lines produced, as O C, lay off a fourth proportional to *m*, *n*, and O C, and through its extremity F, draw *a* parallel to B C, to meet O B produced in G.

Draw *b* parallel to C D, to meet O D produced, and so on.

Then is P′ the required polygon.

For, the sides of P′ being respectively parallel to those of P, the angles of P′ are equal to the corresponding angles of P,

Ch. II, Th. XI.

Also, by similar triangles,

a : B C :: O G : O B,

and *c* : A B :: O G : O B;

∴ *a* : B C :: *c* : A B.

Similarly, *c* : A B :: *e* : A E, and so on.

Hence, the sides of P′ are proportional to the homologous sides of P.

Hence, P′ and P are similar.

Again, by similar triangles,

a : B C :: O F : O C.

But, O F : O C :: *n* : *m*.

∴ *a* : B C :: *n* : *m*, or

B C : *a* :: *m* : *n*. *Q. E. F.*

Scholium.—The given ratio is called the *ratio of similitude;* and the point, O, once assumed, is called the *centre of similitude.*

QUERY.

Why does the construction of the fourth proportional require you to *produce* O C?

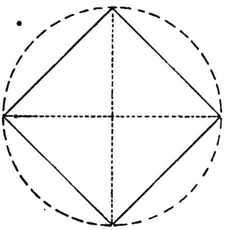
REGULAR POLYGONS.

DEFINITION.

A **Regular Polygon** is one which is both equilateral and equiangular.

PROBLEM XXII.

To inscribe a square in a given circle.

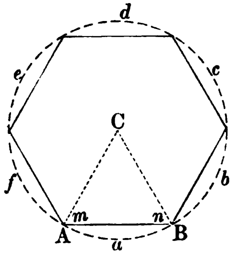


Draw two diameters perpendicular to each other, and join their extremities by chords. These chords form the required square.

[We leave the *proof* as an exercise for the student. *See Ch. III, Ths. I, IV; Th. XX, Cor. II.*]

PROBLEM XXIII.

To inscribe a regular hexagon in a given circle.



Suppose the construction made.

Draw radii to the vertices of any two adjacent angles, as A and B.

The arcs a, b, c, d, e, and f, being subtended by equal chords, are equal.

Hence $a = \frac{1}{6}$ of the circumference $= 60°$.

But angle $C = a$;

$\therefore$ angle $C = 60°$.

But A C B is an isosceles triangle, since $AC = BC$, being radii of the circle.

$\therefore$ $m = 60°$, and $n = 60°$ - - - - *Ch. II, Th. VI.*

$\therefore$ A C B is equiangular.

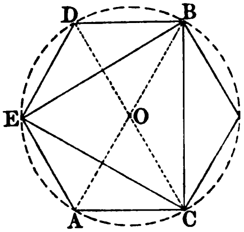
$\therefore$ A C B is equilateral - - - *Ch. II, Th. XXII, Cor.*

Hence, $AB = AC$; that is, any side of a regular inscribed hexagon is equal to the radius of the circle.

Therefore, to inscribe a regular hexagon in a circle, *apply the radius to the circumference six times as a chord.*

Q. E. F.

EXERCISES.



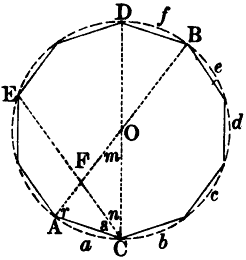
1. Show that A B and C D are diameters.
2. How many degrees in any angle of a regular inscribed hexagon?
3. Prove that A O D E is a rhombus.
4. Prove that E B C is equilateral; and thence state, in general terms, a method of inscribing an equilateral triangle in a given circle.
5. From the right triangle A E B, show that E B = A O $\sqrt{3}$, and state the relation as a principle.
6. Express, as a principle, the relation in the following proportion:

$$EB : AO :: \sqrt{3} : 1.$$

Whence this proportion?

PROBLEM XXIV.

To inscribe a regular decagon in a given circle.



Suppose the construction made.

The sides of the decagon being equal, the subtended arcs are equal.

Hence, any arc, as a, $= \frac{1}{10}$ the circumference $= 36°$

Draw A B and C D, which will be diameters of the circle; for they each bisect the circumference, since

$$a + b + c + d + e = 180°, \text{ and } b + c + d + e + f = 180°.$$

Draw C E, cutting A O in F.

Now, $m = a = 36°$ - - - - *Ch. IV, Th. XIX, Sch. 2.*

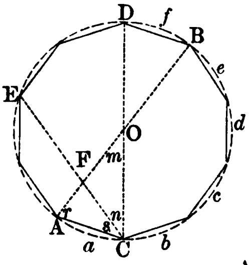
$n = \frac{1}{2} \overset{\frown}{ED} = 36°$ - - - - - - *Ch. IV, Th. XX.*

$\therefore$ O C F is isosceles.

Again, $s = \frac{1}{2} \underset{\smile}{AE} = 36°$.

$\therefore \quad n = s,$

$\therefore \quad m = s.$



Hence, the triangles A O C and A C F have $m = s$, and r common; they are therefore similar (*Ch. VIII, Th. I, Cor. I*).

But A O C being isosceles, A C F is isosceles; and

$$AC = CF = OF.$$

Since A O C and A C F are similar,

$$AO : AC :: AC : AF.$$

But $$AC = OF;$$

$$\therefore \quad AO : OF :: OF : AF.$$

Hence, A O is divided in extreme and mean ratio at F.

Since A C, one side of the assumed decagon, equals O F, the greater segment of A O, we have the following construction for inscribing a regular decagon in a given circle:

Divide the radius of the circle in extreme and mean ratio (*Prob. VII*), and apply the greater segment to the circumference ten times as a chord. *Q. E. F.*

EXERCISES.

1. In the triangle A O C, show that either of the angles at the base is double the angle at O.

2. How many degrees in m?

3. Join the alternate vertices of a regular inscribed decagon, and prove that the resulting figure is a regular inscribed pentagon.

4. If the arcs subtended by the sides of a regular inscribed decagon be bisected, and chords of the semi-arcs be drawn, how many sides will the resulting polygon have? On what principle will it be equilateral? On what principle will it be equiangular?

5. To inscribe in a given circle a regular dodecagon.

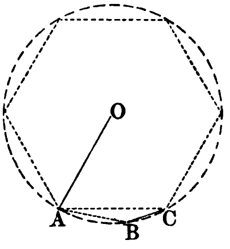
Suggestion.—If you bisect the arcs of the regular inscribed hexagon, into how many equal parts will you divide the circumference?

6. If you bisect the arcs of an inscribed square, and draw chords of the semi-arcs, what will the resulting figure be?

7. In general, any regular inscribed polygon being given, how will you obtain a regular inscribed polygon of twice the number of sides? of half?

PROBLEM XXV.

To inscribe in a given circle a regular pentedecagon.



Let O be the given circle, in which it is required to inscribe a regular pentedecagon.

Construct a regular inscribed hexagon, one of whose sides is A C (*Prob. XXIII*); also, construct a regular inscribed decagon, one of whose sides is A B (*Prob. XXIV*). Join B and C; B C will be the side of the regular inscribed pentedecagon required.

For, $\underset{\smile}{AC} = \frac{1}{6}$ the circumference,

and $\underset{\smile}{AB} = \frac{1}{10}$ the circumference.

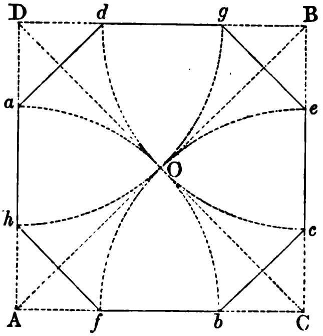
$\underset{\smile}{AC} - \underset{\smile}{AB} = \frac{1}{6} - \frac{1}{10} = \frac{1}{15}$ the circumference.

$\therefore \quad \underset{\smile}{BC} = \frac{1}{15}$ the circumference.

Hence, the chord B C applied 15 times to the circumference will form the pentedecagon required. It is a pentedecagon because it has 15 sides; and it is a regular pentedecagon because its sides are equal chords of the circumference, its angles being equal because they are inscribed in equal segments, or are measured by the halves of equal arcs.

PROBLEM XXVI.

In a given square, to inscribe a regular octagon.



Draw the diagonals A B and C D.

From A, B, C, and D as centres, with radius equal to the semi-diagonal A O, describe arcs cutting the sides in *a* and *b*, *c* and *d*, *e* and *f*, *g* and *h*.

Join the adjacent points of intersection, and the resulting figure will be the octagon required.

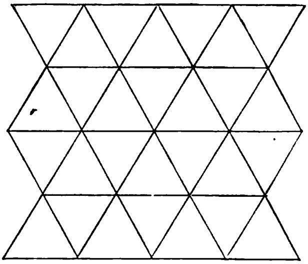
[The *proof* of the construction is left as an exercise for the student. He is to prove the octagon equilateral and equiangular.]

Note.—A carpenter, wishing to 'eight-square' a hand-rail, draws the diagonals A B and C D upon the end of it, after it has been 'squared up.' He then sets a 'gauge' to the distance A *b*, or the semidiagonal, and runs it upon the whole length of the piece from each corner both ways. This shows him how much is to be 'chamfered' off, to make the piece octagonal.

APPLICATIONS.

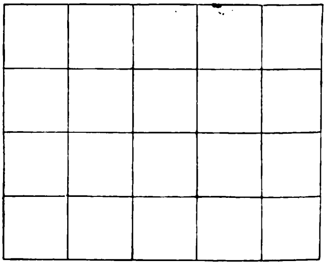
Among the innumerable varieties of form presented by polygons, those which are *regular* stand preëminent, for both beauty and use. From the principles hitherto established, a most noticeable practical inference is, that only *three* regular figures can, by repetition, completely cover a plane surface,—the equilateral triangle, the square, and the regular hexagon.

To cover a surface entirely with a mosaic work of equal polygons, they must be such that their angles, when placed with their vertices coincident, will exactly fill the space surrounding a point.



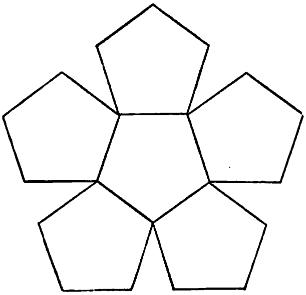
Each angle of an equilateral triangle is 60°, and six such angles = 360°.

Hence, by the conjunction of six equilateral triangles, all the angular space about a point would be occupied.



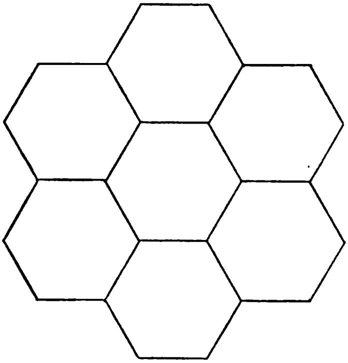
Each angle of a square is 90°, and four such = 360°.

Hence, four squares, when placed in contact, with their vertices coincident, will exactly fill the space surrounding a point, and a plane may be completely covered by equal squares.



Each angle of a regular pentagon is 108°; three such = 324°, and four such = 432°.

Hence, the angles of a regular pentagon cannot be arranged about a point so as to make four right angles.



Each angle of a regular hexagon is 120°, and three such = 360°.

Hence, a plane may be covered by a combination of regular hexagons.

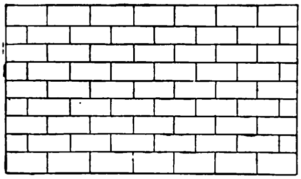
The bee — that instinct mathematician — makes use of this fact in building its honeycomb.

Since the angles increase in magnitude, when the sides increase in number, three angles of any regular polygon of more than six sides must be greater than 360°; and hence no other regular forms can be employed for the purpose proposed.

QUERIES.

1. Which is the better suited to support an incumbent weight or resist a lateral force, a combination of equilateral triangles or a combination of regular hexagons? Does the conjunction of angles at the same point weaken the surface and increase the liability to fracture?

2. Why do masons adopt the arrangement in the annexed diagram?



EXERCISES.

1. Draw a line, and divide it into parts proportional to 3, 7, 5, and 8.
2. To trisect a given straight line.

3. Three lines are 5, 3, and 7: find their fourth proportional.

4. Two lines are 11 and 9; find their third proportional.

5. Two lines are 10½ and 4; construct their mean proportional.

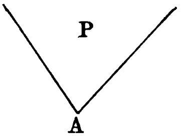
6. Given the proportions, $3 : 8 :: 5 : x$, and $8 : 3 :: 5 : x$, to construct x.

7. Given $x = \sqrt{7}$, to construct x.

Suggestion. $x = \sqrt{7} = \sqrt{7 \times 1}$.

8. To divide a line 10 inches long internally in extreme and mean ratio. (*See Prob. VII.*)

9. To construct a triangle whose sides are 7, 9, and 11; then to construct a similar triangle whose side homologous to side 9, is 5. (*See Prob. X.*)



10. Through the point P, in the angle A, to draw a line, so that the segments between the point and the sides of the angle shall be equal. (*See Prob. IX.*)

11. To construct a triangle equal to a given hexagon.

12. The area of a square is 16; to construct a square that shall be to it in the ratio of 5 to 3. (*See Prob. XVII.*)

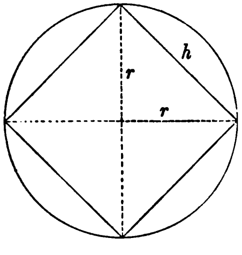
13. To construct a square that shall be equal to a triangle whose area is 32, and base 8.

14. To construct a square that shall be equal to a given pentagon plus a given parallelogram.

Suggestion.—Find squares separately equal to the pentagon and parallelogram. (*See Probs. XI, XII, and XIII.*)

15. The areas of two squares are 25 and 16; to construct the square equal to their difference. (*See Prob. XIV.*)

16. The area of a rectangle is 48, and the base 8; to construct on the base 12½ an equal rectangle.



17. A pentagon being given, to construct a similar pentagon whose sides shall be to those of the given pentagon in the ratio of 3 to 5.

Suggestion.—Observe the order of the terms. (*See Prob. XXI.*)

18. The radius of a circle is to the side of the inscribed square as 1 is to the square root of 2.

19. To construct an isosceles triangle having either of the angles at the base twice the third angle.

Suggestion.—Draw a line, and divide it in extreme and mean ratio. (*See Prob. XXIV.*)

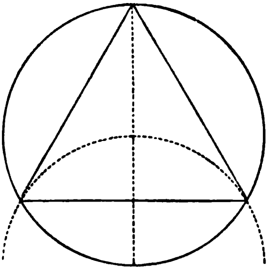
20. A hexagon being given, to construct a similar one whose area shall be in the ratio of 5 to 3 to that of the given one.

21. The area of a square is 16; to find the length of a side of the regular inscribed octagon.

22. Given two pentagons, to construct a square equal to their difference.

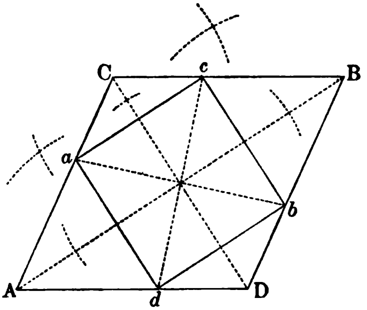
23. To inscribe in a circle an equilateral triangle.

Suggestion.—Draw a diameter, and from extremity of it as centre, with radius of given circle, describe an arc cutting given circumference. Join points of intersection with the other extremity of the diameter. Prove the triangle to be equiangular.



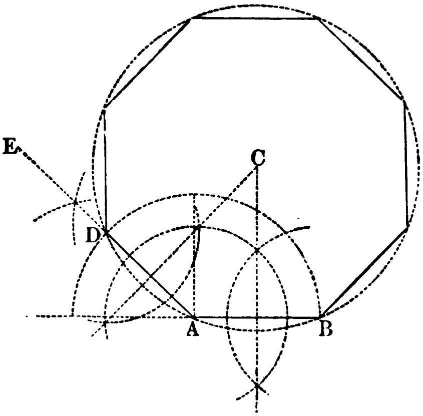
24. To inscribe a square in a given rhombus.

Suggestion.—A B and C D are diagonals; *a b* and *c d*, bisectors of angles. How do the diagonals of a rhombus intersect each other?

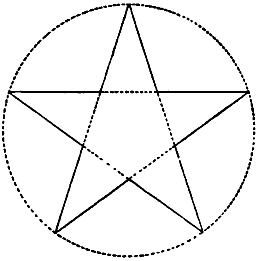


25. On a given base, to construct a regular octagon.

Suggestion.—Let A B be the given base. Produce it, and from A as centre, with radius = A B, describe a semicircle. Divide the semicircumference into four equal parts. To the first point of division, draw A D, and pass a circumference through D, A, and B, *etc.*



26. On a given base, to construct a regular pentagon.



Suggestion.—General method same as in the preceding. Divide the radius in extreme and mean ratio, and thence divide the semicircumference into five equal parts. How many degrees in an interior angle of a regular pentagon?

27. To construct a five-point star.

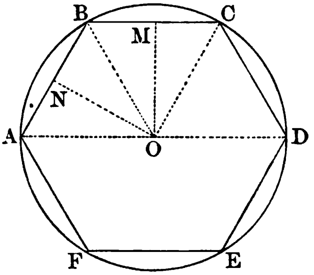
Suggestion.—Divide the circumference of circle into five equal parts.

CHAPTER X.

REGULAR POLYGONS AND CIRCLES.

THEOREM I.

A circumference may be circumscribed about any regular polygon.



Let A B C D E F be a regular polygon; then may a circumference be circumscribed about it.

For, bisect A B and B C by the perpendiculars O N and O M.

From centre O, with radius O B, pass a circumference through A, B, and C - *Ch. IV, Th. IX.*

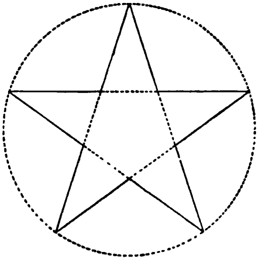
Let A O M B be revolved about O M (*folded over*).

Since the angles at M are right, and M B = M C, B will coincide with C.

Since the angle B = the angle C, and A B = C D, A will coincide with D.

Hence, the circumference which passes through A, B, and C, will pass through D.

Similarly, the circumference passing through B, C, and D, will pass through E, and thus through all the vertices of the polygon; and hence is circumscribed. *Q. E. D.*



Suggestion.—General method same as in the preceding. Divide the radius in extreme and mean ratio, and thence divide the semicircumference into five equal parts. How many degrees in an interior angle of a regular pentagon?

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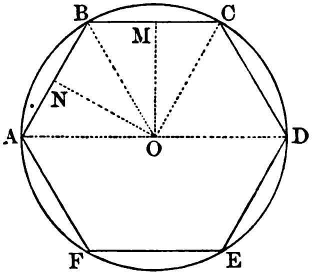


Let A B C D E F be a regular polygon; then may a circumference be circumscribed about it.

For, bisect A B and B C by the perpendiculars O N and O M.

From centre O, with radius O B, pass a circumference through A, B, and C - *Ch. IV, Th. IX.*

Let A O M B be revolved about O M (*folded over*).

Since the angles at M are right, and M B = M C, B will coincide with C.

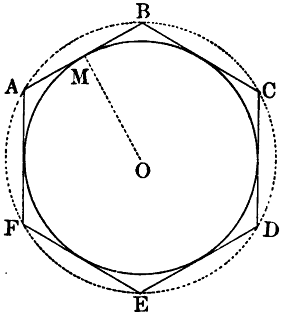
Since the angle B = the angle C, and A B = C D, A will coincide with D.

Hence, the circumference which passes through A, B, and C, will pass through D.

Similarly, the circumference passing through B, C, and D, will pass through E, and thus through all the vertices of the polygon; and hence is circumscribed. *Q. E. D.*

THEOREM II.

A circumference may be inscribed in any regular polygon.



Let A B C D E F be any regular polygon; then may a circumference be inscribed in it.

For, let a circumference be circumscribed about it - - - - - *Th. I.*

The sides of the polygon, being equal, are equal chords of the circumscribed circle and hence are equally distant from the centre - - - - - - - - *Ch. IV, Th. III.*

Hence, the circumference described from centre O, with radius = O M, will touch all the sides of the polygon, and is therefore inscribed. *Q. E. D.*

DEFINITIONS.

From Theorems I and II we learn, that within a regular polygon there is a point equally distant from all the vertices,—the centre of the circumscribed circle; also, that the *same* point is equally distant from the sides,—the centre of the inscribed circle; and that this point is hence the common centre of the circumscribed and inscribed circles. These considerations make clear the following definitions:

1. The **Centre** of a regular polygon is the common centre of the circumscribed and inscribed circles.

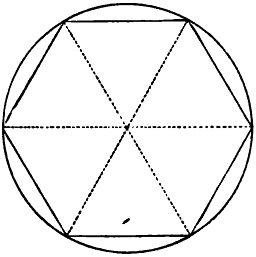
2. The **Radius** of a regular polygon is the radius of the circumscribed circle.

3. The **Apothem** of a regular polygon is the radius of the inscribed circle.

4. A **Central Angle** of a regular polygon is an angle included between two radii drawn to the extremities of the same side.

QUERIES.

1. Are all the central angles of a regular polygon equal, each to each? Why?



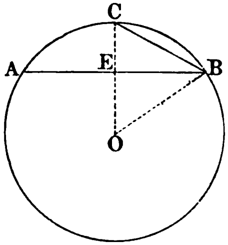
2. What is the general expression, in right angles, for any central angle of a regular polygon? *Ans.* $\frac{4\,R}{n}$, *n* being the number of sides. (*See Ch. II, Th. II, Cor.*)

3. How may you find the centre of a regular polygon?—Bisect, by perpendiculars, any two adjacent sides, or bisect any two adjacent angles.

4. How is the apothem situated with respect to the sides? (*See diagrams of Ths. I, II.*)

PROBLEM.

Given the chord of an arc, in a circle whose radius is 1, to find the chord of one-half that arc.



Let O be the circle; and A B the given chord.

Draw the radii O B and O C, O C perpendicular to A B.

Then $\overset{\frown}{BC} = \overset{\frown}{AC} = \frac{1}{2}\,\overset{\frown}{AB}$.

Denote the given chord A B by 2 *c*, and the required chord B C by *x*.

$$\overline{OE}^2 = r^2 - c^2;\ \therefore\ OE = \sqrt{r^2 - c^2}$$

But, $$r = 1;\ \therefore\ OE = \sqrt{1 - c^2}.$$

$$x^2 = \overline{EC}^2 + c^2;\ \therefore\ x = \sqrt{\overline{EC}^2 + c^2}.$$

But, $$EC = OC - OE = 1 - \sqrt{1 - c^2};$$

$$\therefore\ \overline{EC}^2 = 1 - 2\sqrt{1 - c^2} + 1 - c^2 = 2 - 2\sqrt{1 - c^2} - c^2$$

Hence, $$x = \sqrt{2 - 2\sqrt{1 - c^2} - c^2 + c^2} = \sqrt{2 - 2\sqrt{1 - c^2}}.$$

That is, in a circle whose radius is 1, *any* chord being *known* or *found*, the chord of half its arc may be found by subtracting

from 2, *twice the square root of* (1 — the square of half the given chord), then extracting the square root of the result. *Q. E. F.*

Note.—The student should understand clearly that c and x are *general;* c representing one-half of *any known* chord in a circle whose radius is *unity*, and x representing the chord of half the arc. Therefore, having found 'the chord of half the arc,' this value of x, being known, becomes $2c$; and we may use it in the preceding formula to find the chord of half of the half, then the half of this half, and so on indefinitely, to chords of arcs inconceivably small.

THEOREM III.

The circumference of a circle whose radius is 1, *is* 6.28318, *nearly.*

We shall approximate the circumference of a circle whose radius is 1, by first obtaining the chord of 60°, then the chord of half this arc, then the chord of half of the half, and so on, till there is no appreciable difference between the length of the chord and that of its subtended arc.

The chord of 60°, being the side of a regular inscribed hexagon, is equal to the radius. - - - - *Ch. IX, Prob. XXIII.*

But, in the present instance, the radius = 1.

Hence, the chord of 60° = 1.

Now starting with this chord as given, $2c$ (which denotes the given chord), will equal 1; whence, in the formula

$$x = \sqrt{2 - 2\sqrt{1 - c^2}},\ c \text{ will equal } \tfrac{1}{2} \text{ and } c^2 = \tfrac{1}{4}.$$

$$\therefore\quad x = \sqrt{2 - 2\sqrt{1 - \tfrac{1}{4}}} = \sqrt{2 - 2\sqrt{\tfrac{3}{4}}} = \sqrt{2 - \sqrt{3}}.$$

$$= \sqrt{2 - 1.732050805} = \sqrt{.267949195}$$

$= .5176380902 =$ the chord of half the arc $=$ the chord of 30°.

Having found the chord of 30°, we may similarly find the chord of 15°; $2c$ being .5176380902, and c one-half as much.

Continuing this process by successive applications of the general formula, the results will be found as in the following

TABLE.

Chord of	60°	(= $\frac{1}{6}$	of the perimeter)	= 1.0000000000.
"	30°	(= $\frac{1}{12}$	" ")	= .5176380902.
"	15°	(= $\frac{1}{24}$	" ")	= .2610523842.
"	7° 30′	(= $\frac{1}{48}$	" ")	= .1308062583.
"	3° 45′	(= $\frac{1}{96}$	" ")	= .0654381655.
"	1° 52′ 30″	(= $\frac{1}{192}$	" ")	= .0327234632.
"	56′ 15″	(= $\frac{1}{384}$	" ")	= .0163622792.
"	28′ 7½″	(= $\frac{1}{768}$	" ")	= .0081812080.
"	14′ 3¾″	(= $\frac{1}{1536}$	" ")	= .0040906112.
"	7′ 1⅞″	(= $\frac{1}{3072}$	" ")	= .0020453068.

Now, it is evident, between an arc so minute as 7 minutes of a degree, and its chord, there can be no appreciable difference; and hence the length of the chord may be taken as the length of the subtended arc. If, then, we take .0020453068 to be the linear value of $\frac{1}{3072}$ of the entire circumference,

Circumference whose radius is unity

$$= .0020453068 \times 3072$$
$$= 6.283183104 = 6.28318 +.$$

Cor.—The length of the semicircumference of a circle whose radius is 1, is 3.14159 +, used practically as 3.1416.

Scholium 1.—The linear value 3.1416 is usually denoted by the Greek letter π, pronounced *pi*, and signifying, primarily, *the semicircumference of a circle whose radius is unity*, or 180° of the circumference. 2π may therefore be adopted as the symbol for 360°; $\frac{1}{2}\pi$ as the symbol for 90°; $\frac{1}{4}\pi$ as the symbol for 45°, *etc*; the radius being always supposed 1, unless the contrary is stated.

Scholium 2.—Finding the linear value of a circumference (that is, finding a straight line equal to it in length), is known as the rectification of the circumference. It has been a celebrated problem from the time of Archimedes, but the various attempts have resulted in only an approximate solution, which, however, is so close that the exact solution is not a question of any practical importance.

[The *proof* of the construction is left as an exercise for the student. He is to prove the octagon equilateral and equiangular.]

Note.—A carpenter, wishing to 'eight-square' a hand-rail, draws the diagonals A B and C D upon the end of it, after it has been 'squared up.' He then sets a 'gauge' to the distance A *b*, or the semidiagonal, and runs it upon the whole length of the piece from each corner both ways. This shows him how much is to be 'chamfered' off, to make the piece octagonal.

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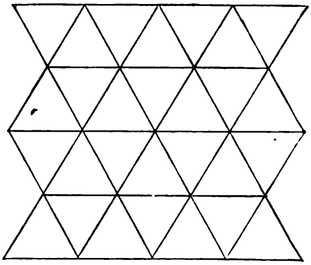
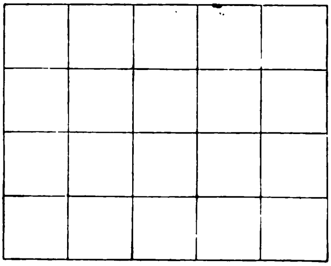
Among the innumerable varieties of form presented by polygons, those which are *regular* stand preëminent, for both beauty and use. From the principles hitherto established, a most noticeable practical inference is, that only *three* regular figures can, by repetition, completely cover a plane surface,—the equilateral triangle, the square, and the regular hexagon.

To cover a surface entirely with a mosaic work of equal polygons, they must be such that their angles, when placed with their vertices coincident, will exactly fill the space surrounding a point.



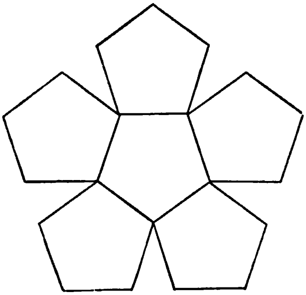
Each angle of an equilateral triangle is 60°, and six such angles = 360°.

Hence, by the conjunction of six equilateral triangles, all the angular space about a point would be occupied.



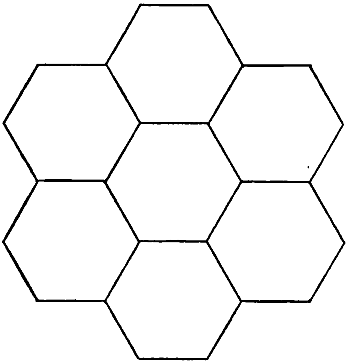
Each angle of a square is 90°, and four such = 360°.

Hence, four squares, when placed in contact, with their vertices coincident, will exactly fill the space surrounding a point, and a plane may be completely covered by equal squares.



Each angle of a regular pentagon is 108°; three such = 324°, and four such = 432°.

Hence, the angles of a regular pentagon cannot be arranged about a point so as to make four right angles.



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Hence, a plane may be covered by a combination of regular hexagons.

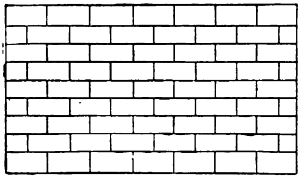
The bee — that instinct mathematician — makes use of this fact in building its honeycomb.

Since the angles increase in magnitude, when the sides increase in number, three angles of any regular polygon of more than six sides must be greater than 360°; and hence no other regular forms can be employed for the purpose proposed.

QUERIES.

1. Which is the better suited to support an incumbent weight or resist a lateral force, a combination of equilateral triangles or a combination of regular hexagons? Does the conjunction of angles at the same point weaken the surface and increase the liability to fracture?

2. Why do masons adopt the arrangement in the annexed diagram?



EXERCISES.

1. Draw a line, and divide it into parts proportional to 3, 7, 5, and 8.
2. To trisect a given straight line.

3. Three lines are 5, 3, and 7: find their fourth proportional.

4. Two lines are 11 and 9; find their third proportional.

5. Two lines are 10½ and 4; construct their mean proportional.

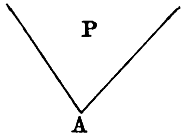
6. Given the proportions, $3 : 8 :: 5 : x$, and $8 : 3 :: 5 : x$, to construct x.

7. Given $x = \sqrt{7}$, to construct x.

Suggestion. $x = \sqrt{7} = \sqrt{7 \times 1}$.

8. To divide a line 10 inches long internally in extreme and mean ratio. (*See Prob. VII.*)

9. To construct a triangle whose sides are 7, 9, and 11; then to construct a similar triangle whose side homologous to side 9, is 5. (*See Prob. X.*)



10. Through the point P, in the angle A, to draw a line, so that the segments between the point and the sides of the angle shall be equal. (*See Prob. IX.*)

11. To construct a triangle equal to a given hexagon.

12. The area of a square is 16; to construct a square that shall be to it in the ratio of 5 to 3. (*See Prob. XVII.*)

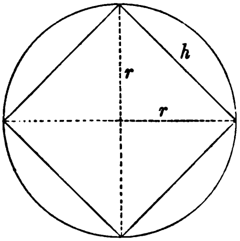
13. To construct a square that shall be equal to a triangle whose area is 32, and base 8.

14. To construct a square that shall be equal to a given pentagon plus a given parallelogram.

Suggestion.—Find squares separately equal to the pentagon and parallelogram. (*See Probs. XI, XII, and XIII.*)

15. The areas of two squares are 25 and 16; to construct the square equal to their difference. (*See Prob. XIV.*)

16. The area of a rectangle is 48, and the base 8; to construct on the base 12½ an equal rectangle.



17. A pentagon being given, to construct a similar pentagon whose sides shall be to those of the given pentagon in the ratio of 3 to 5.

Suggestion.—Observe the order of the terms. (*See Prob. XXI.*)

18. The radius of a circle is to the side of the inscribed square as 1 is to the square root of 2.

19. To construct an isosceles triangle having either of the angles at the base twice the third angle.

Suggestion.—Draw a line, and divide it in extreme and mean ratio. (*See Prob. XXIV.*)

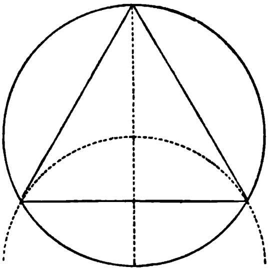
20. A hexagon being given, to construct a similar one whose area shall be in the ratio of 5 to 3 to that of the given one.

21. The area of a square is 16; to find the length of a side of the regular inscribed octagon.

22. Given two pentagons, to construct a square equal to their difference.

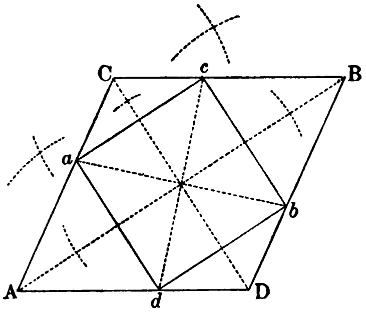
23. To inscribe in a circle an equilateral triangle.

Suggestion.—Draw a diameter, and from extremity of it as centre, with radius of given circle, describe an arc cutting given circumference. Join points of intersection with the other extremity of the diameter. Prove the triangle to be equiangular.



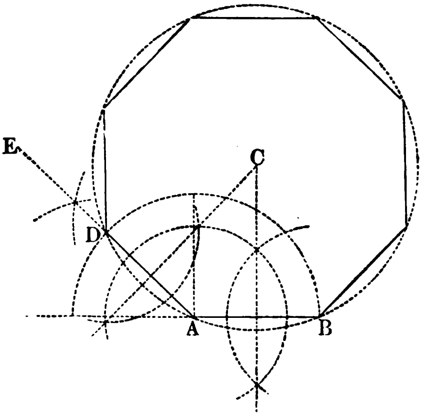
24. To inscribe a square in a given rhombus.

Suggestion.—A B and C D are diagonals; *a b* and *c d*, bisectors of angles. How do the diagonals of a rhombus intersect each other?

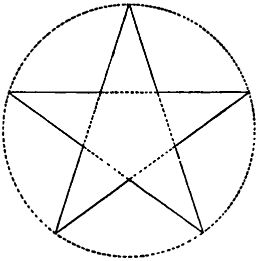


25. On a given base, to construct a regular octagon.

Suggestion.—Let A B be the given base. Produce it, and from A as centre, with radius = A B, describe a semicircle. Divide the semicircumference into four equal parts. To the first point of division, draw A D, and pass a circumference through D, A, and B, *etc.*



26. On a given base, to construct a regular pentagon.



Suggestion.—General method same as in the preceding. Divide the radius in extreme and mean ratio, and thence divide the semicircumference into five equal parts. How many degrees in an interior angle of a regular pentagon?

27. To construct a five-point star.

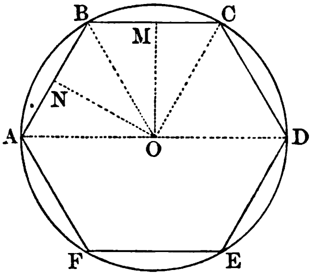
Suggestion.—Divide the circumference of circle into five equal parts.

CHAPTER X.

REGULAR POLYGONS AND CIRCLES.

THEOREM I.

A circumference may be circumscribed about any regular polygon.



Let A B C D E F be a regular polygon; then may a circumference be circumscribed about it.

For, bisect A B and B C by the perpendiculars O N and O M.

From centre O, with radius O B, pass a circumference through A, B, and C - *Ch. IV, Th. IX.*

Let A O M B be revolved about O M (*folded over*).

Since the angles at M are right, and M B = M C, B will coincide with C.

Since the angle B = the angle C, and A B = C D, A will coincide with D.

Hence, the circumference which passes through A, B, and C, will pass through D.

Similarly, the circumference passing through B, C, and D, will pass through E, and thus through all the vertices of the polygon; and hence is circumscribed. *Q. E. D.*

THEOREM II.

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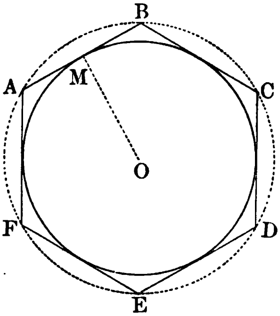

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For, let a circumference be circumscribed about it - - - - - *Th. I.*

The sides of the polygon, being equal, are equal chords of the circumscribed circle and hence are equally distant from the centre - - - - - - - - *Ch. IV, Th. III.*

Hence, the circumference described from centre O, with radius = O M, will touch all the sides of the polygon, and is therefore inscribed. *Q. E. D.*

DEFINITIONS.

From Theorems I and II we learn, that within a regular polygon there is a point equally distant from all the vertices,—the centre of the circumscribed circle; also, that the *same* point is equally distant from the sides,—the centre of the inscribed circle; and that this point is hence the common centre of the circumscribed and inscribed circles. These considerations make clear the following definitions:

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2. The **Radius** of a regular polygon is the radius of the circumscribed circle.

3. The **Apothem** of a regular polygon is the radius of the inscribed circle.

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QUERIES.

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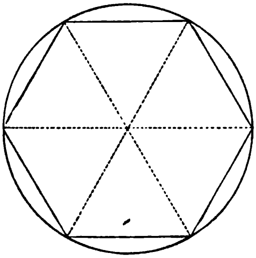

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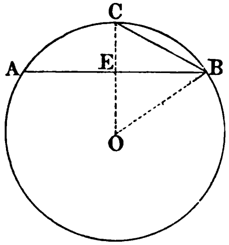

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Denote the given chord A B by $2\,c$, and the required chord B C by x.

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$$x^2 = \overline{\mathrm{E\,C}}^2 + c^2;\ \therefore\ x = \sqrt{\mathrm{E\,C}^2 + c^2}.$$

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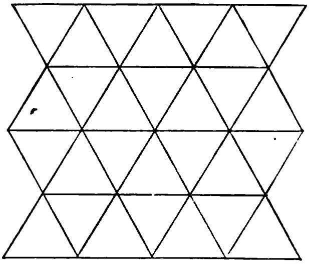

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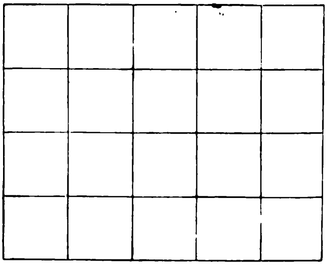

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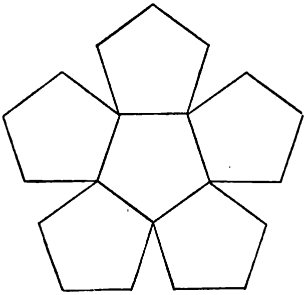

Each angle of a regular pentagon is 108°; three such = 324°, and four such = 432°.

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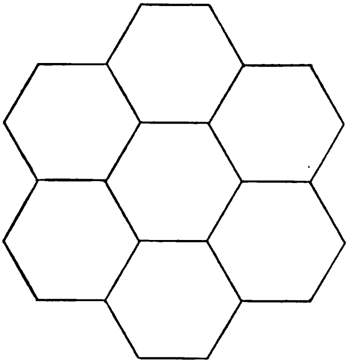

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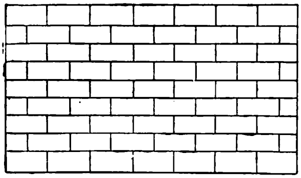

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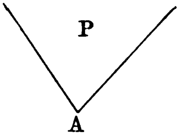
9. To construct a triangle whose sides are 7, 9, and 11; then to construct a similar triangle whose side homologous to side 9, is 5. (*See Prob. X.*)



10. Through the point P, in the angle A, to draw a line, so that the segments between the point and the sides of the angle shall be equal. (*See Prob. IX.*)

11. To construct a triangle equal to a given hexagon.

12. The area of a square is 16; to construct a square that shall be to it in the ratio of 5 to 3. (*See Prob. XVII.*)

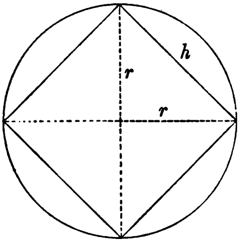
13. To construct a square that shall be equal to a triangle whose area is 32, and base 8.

14. To construct a square that shall be equal to a given pentagon plus a given parallelogram.

Suggestion.—Find squares separately equal to the pentagon and parallelogram. (*See Probs. XI, XII, and XIII.*)

15. The areas of two squares are 25 and 16; to construct the square equal to their difference. (*See Prob. XIV.*)

16. The area of a rectangle is 48, and the base 8; to construct on the base 12½ an equal rectangle.



17. A pentagon being given, to construct a similar pentagon whose sides shall be to those of the given pentagon in the ratio of 3 to 5.

Suggestion.—Observe the order of the terms. (*See Prob. XXI.*)

18. The radius of a circle is to the side of the inscribed square as 1 is to the square root of 2.

19. To construct an isosceles triangle having either of the angles at the base twice the third angle.

Suggestion.—Draw a line, and divide it in extreme and mean ratio. (*See Prob. XXIV.*)

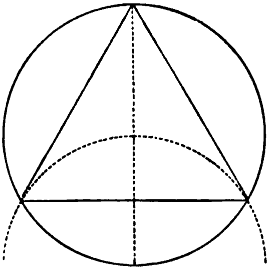
20. A hexagon being given, to construct a similar one whose area shall be in the ratio of 5 to 3 to that of the given one.

21. The area of a square is 16; to find the length of a side of the regular inscribed octagon.

22. Given two pentagons, to construct a square equal to their difference.

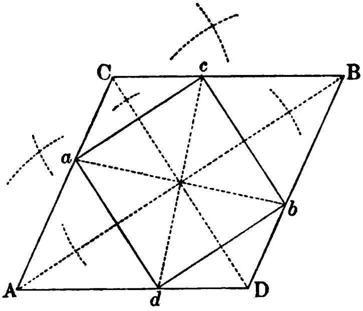
23. To inscribe in a circle an equilateral triangle.

Suggestion.—Draw a diameter, and from extremity of it as centre, with radius of given circle, describe an arc cutting given circumference. Join points of intersection with the other extremity of the diameter. Prove the triangle to be equiangular.



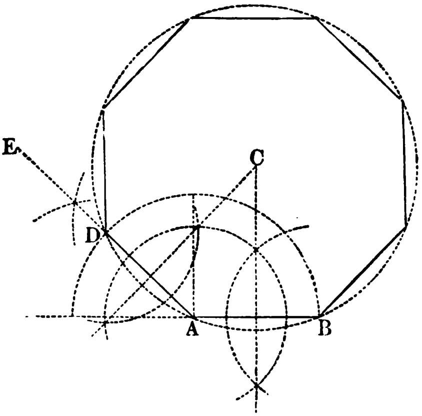
24. To inscribe a square in a given rhombus.

Suggestion.—A B and C D are diagonals; *a b* and *c d*, bisectors of angles. How do the diagonals of a rhombus intersect each other?

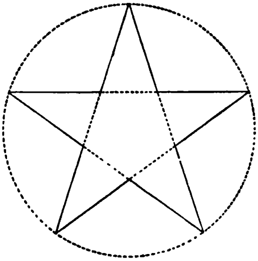


25. On a given base, to construct a regular octagon.

Suggestion.—Let A B be the given base. Produce it, and from A as centre, with radius = A B, describe a semicircle. Divide the semicircumference into four equal parts. To the first point of division, draw A D, and pass a circumference through D, A, and B, *etc.*



26. On a given base, to construct a regular pentagon.



Suggestion.—General method same as in the preceding. Divide the radius in extreme and mean ratio, and thence divide the semicircumference into five equal parts. How many degrees in an interior angle of a regular pentagon?

27. To construct a five-point star.

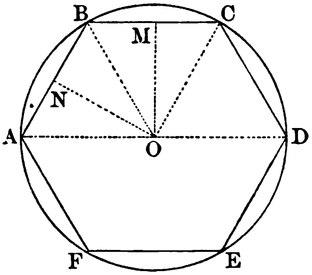
Suggestion.—Divide the circumference of circle into five equal parts.

CHAPTER X.

REGULAR POLYGONS AND CIRCLES.

THEOREM I.

A circumference may be circumscribed about any regular polygon.



Let A B C D E F be a regular polygon; then may a circumference be circumscribed about it.

For, bisect A B and B C by the perpendiculars O N and O M.

From centre O, with radius O B, pass a circumference through A, B, and C - *Ch. IV, Th. IX.*

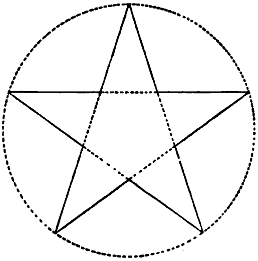
Let A O M B be revolved about O M (*folded over*).

Since the angles at M are right, and M B = M C, B will coincide with C.

Since the angle B = the angle C, and A B = C D, A will coincide with D.

Hence, the circumference which passes through A, B, and C, will pass through D.

Similarly, the circumference passing through B, C, and D, will pass through E, and thus through all the vertices of the polygon; and hence is circumscribed. *Q. E. D.*



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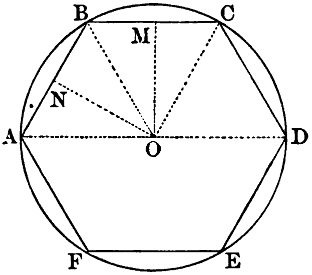

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For, bisect A B and B C by the perpendiculars O N and O M.

From centre O, with radius O B, pass a circumference through A, B, and C - *Ch. IV, Th. IX.*

Let A O M B be revolved about O M (*folded over*).

Since the angles at M are right, and M B = M C, B will coincide with C.

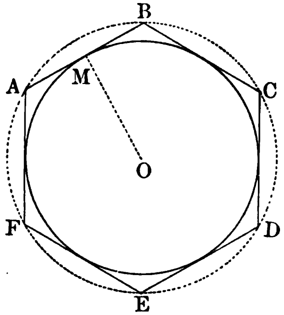
Since the angle B = the angle C, and A B = C D, A will coincide with D.

Hence, the circumference which passes through A, B, and C, will pass through D.

Similarly, the circumference passing through B, C, and D, will pass through E, and thus through all the vertices of the polygon; and hence is circumscribed. *Q. E. D.*

THEOREM II.

A circumference may be inscribed in any regular polygon.



Let A B C D E F be any regular polygon; then may a circumference be inscribed in it.

For, let a circumference be circumscribed about it - - - - - *Th. I.*

The sides of the polygon, being equal, are equal chords of the circumscribed circle and hence are equally distant from the centre - - - - - - - - *Ch. IV, Th. III.*

Hence, the circumference described from centre O, with radius = O M, will touch all the sides of the polygon, and is therefore inscribed. *Q. E. D.*

DEFINITIONS.

From Theorems I and II we learn, that within a regular polygon there is a point equally distant from all the vertices,—the centre of the circumscribed circle; also, that the *same* point is equally distant from the sides,—the centre of the inscribed circle; and that this point is hence the common centre of the circumscribed and inscribed circles. These considerations make clear the following definitions:

1. The **Centre** of a regular polygon is the common centre of the circumscribed and inscribed circles.

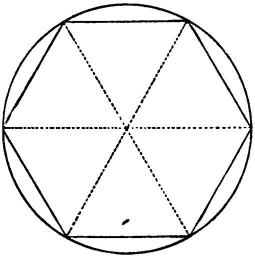
2. The **Radius** of a regular polygon is the radius of the circumscribed circle.

3. The **Apothem** of a regular polygon is the radius of the inscribed circle.

4. A **Central Angle** of a regular polygon is an angle included between two radii drawn to the extremities of the same side.

QUERIES.

1. Are all the central angles of a regular polygon equal, each to each? Why?



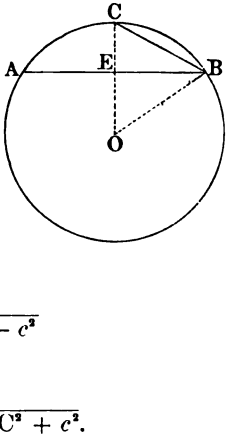
2. What is the general expression, in right angles, for any central angle of a regular polygon? *Ans.* $\frac{4\,R}{n}$, n being the number of sides. (*See Ch. II, Th. II, Cor.*)

3. How may you find the centre of a regular polygon?—Bisect, by perpendiculars, any two adjacent sides, or bisect any two adjacent angles.

4. How is the apothem situated with respect to the sides? (*See diagrams of Ths. I, II.*)

PROBLEM.

Given the chord of an arc, in a circle whose radius is 1, to find the chord of one-half that arc.



Let O be the circle; and A B the given chord.

Draw the radii O B and O C, O C perpendicular to A B.

Then $\overset{\frown}{BC} = \overset{\frown}{AC} = \frac{1}{2}\,\overset{\frown}{AB}$.

Denote the given chord A B by 2 c, and the required chord B C by x.

$$\overline{OE}^2 = r^2 - c^2;\ \therefore\ OE = \sqrt{r^2 - c^2}$$

But, $$r = 1;\ \therefore\ OE = \sqrt{1 - c^2}.$$

$$x^2 = \overline{EC}^2 + c^2;\ \therefore\ x = \sqrt{\overline{EC}^2 + c^2}.$$

But, $$EC = OC - OE = 1 - \sqrt{1 - c^2};$$

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Hence, $$x = \sqrt{2 - 2\sqrt{1 - c^2} - c^2 + c^2} = \sqrt{2 - 2\sqrt{1 - c^2}}.$$

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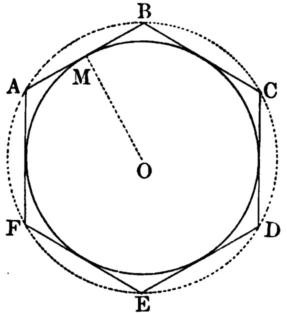

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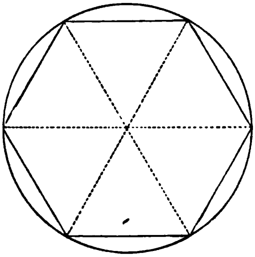

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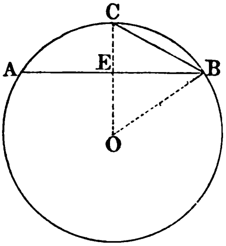

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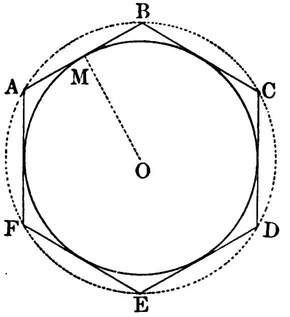

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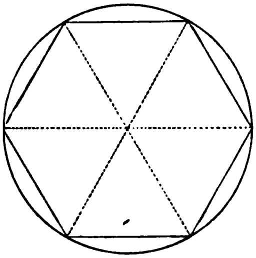

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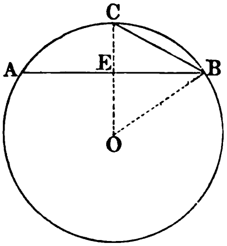

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That is, in a circle whose radius is 1, *any* chord being *known* or *found*, the chord of half its arc may be found by subtracting

Note.—Among the properties of regular polygons, remarkable for their beauty, generality, and usefulness, the following should be remembered:—

1. *Of all plane figures having the same number of sides and the same area, that which is regular has the least outline.*
2. *Of all regular figures having a given area, that which has the greatest number of sides has the least outline.*
3. *Of all plane figures having equal outlines, the circle includes the greatest area.*
4. *Of all plane figures containing the same area, the circle has the least perimeter.*

The object being to convey the greatest quantity of fluid with the least expense of materials, circular pipes are preferable for the conveyance of gas, water, *etc.*; because, among other reasons, in a given length, with a given quantity of material, they have the largest capacity.

If a Gothic window, of given dimensions, consists of regular-shaped panes of glass, a less amount of metallic framing is required than with any other form of pane; and this amount is diminished as the sides of the polygonal form become more numerous. The form of pane requiring the least expenditure of materials, is the regular hexagon.

It will be remembered, that to cover a plane surface with regular figures, one of three forms must be used—the equilateral triangle, the square, or the regular hexagon.

Only these three can completely cover the space about a point; the bee chooses one of these, the regular hexagon.

PART II.

GEOMETRY OF SPACE.

CHAPTER XII.

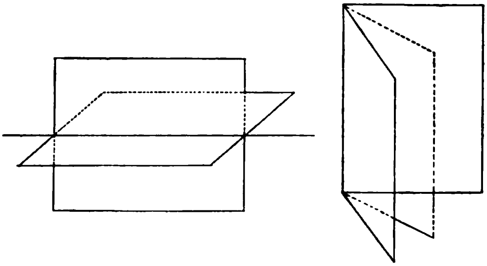
PLANES AND THEIR ANGLES.

DEFINITIONS.

1. A **Plane** (or **Plane Surface**), as already defined, is a surface such that the straight line joining any two of its points lies wholly in the surface.

Note.—To ascertain whether a given surface is a plane, apply the straight edge of a ruler to it, in all directions. If the ruler touches throughout its whole length, the surface is a plane.

2. **Geometry of Space** is that part of geometry which treats of those figures that do not lie wholly in one plane.

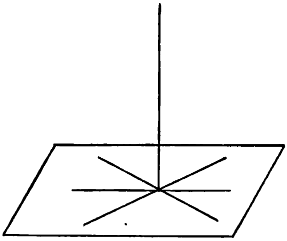


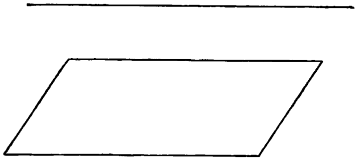
Note.—In the preceding books, we have considered the properties of those geometrical figures which do lie wholly in one plane.

Planes are understood to be of indefinite extent, but limited portions of them are usually represented by parallelograms.

3. **A Perpendicular to a Plane** is a line which is perpendicular to every line of the plane which passes through its foot.

The plane is then perpendicular to the line.



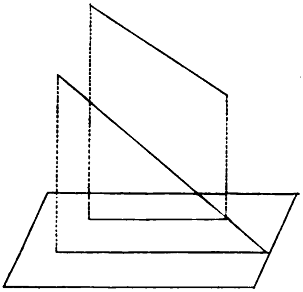


4. A line is **Parallel to a Plane** when the two cannot meet, how far soever both may be produced.

The plane is then parallel to the line.

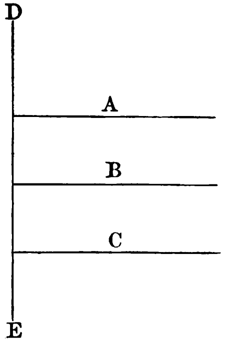
5. A line is **Oblique to a Plane** when it is neither perpendicular nor parallel to the plane.

6. The **Projection of a Point** on a plane is the foot of the perpendicular from the point to the plane.



7. The **Projection of a Line** on a plane is the straight line joining the projections of its extremities.

Note.—If one extremity of the line is in the plane, the perpendicular reduces to zero, and the extremity is its own projection.



8. The **Inclination of a Line** to a plane is the angle which it makes with its projection on the plane.

If the parallel lines A, B, and C be situated in the same plane, and this plane be revolved about the axis E D, to which these lines are perpendicular, they will each describe a plane. During the revolution, these lines will have the same direction, and occupy the same relative position, continually. Hence:

9. Two **Planes are Parallel** when they cannot meet, how far soever both may be produced.

Cor. I.—*If a straight line has two of its points in a plane, it will lie wholly in that plane* (*See Def. 1*).

Cor. II.—*Two parallel planes are everywhere equally distant.*

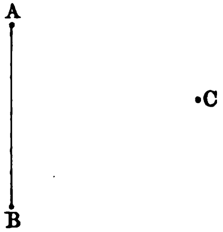
Cor. III.—*Two planes parallel to a third, are parallel to each other.*

Cor. IV.—*Two planes perpendicular to the same line are parallel to each other.*

Cor. V.—*A straight line perpendicular to one of two parallel planes, is perpendicular to the other.*

THEOREM I.

Through three points not in the same straight line, one plane can be passed, and only one.



Let A, B, and C be three points not in the same straight line; then can one plane be passed through them, and only one.

For, join two of the points, as A and B, by the line A B.

Through A B, conceive a plane to be passed, and let it revolve about A B, as an axis, till it contains the point C; it will then pass through the three points, A, B and C.

If now the plane be revolved from this position, either way, it will no longer contain C.

Hence, only this one plane can be passed through the three points. *Q. E. D.*

Cor. I.—*Through two points, or through a line, an indefinite number of planes can be passed.*

Cor. II.—*Three points not in the same straight line, determine the position of a plane.*

Note.—Hence we see that three points, in whatever position placed, must always lie in the same plane. More than three points—four for example—may or may not lie in the same plane, the fourth being above or below the plane which passes through the other three. It is on this principle that a three-legged stool is steady, while a four-legged one may not be.

EXERCISES.

1. Two intersecting lines determine the position of a plane.
2. Two parallel lines determine the position of a plane.
3. The intersection of two planes is a straight line.

Suggestion.—Suppose three common points could be found, which were not in the same straight line? How many planes can embrace three such points?

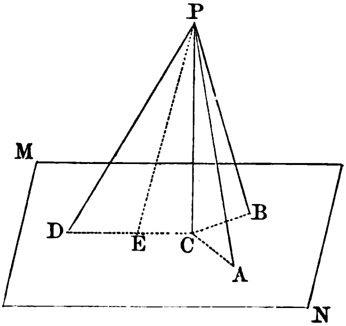
THEOREM II.

If from a point without a plane, a perpendicular and oblique lines be drawn:

1. *The perpendicular will be shorter than any oblique line.*

2. *Oblique lines meeting the plane at equal distances from the foot of the perpendicular, will be equal.*

3. *Of two oblique lines meeting the plane at unequal distances from the foot of the perpendicular, the one which meets it at the greater distance is the longer.*



Let P be the point; M N, the plane; P C, the perpendicular; P A, P B, and P D, the oblique lines; then,

First: P C will be shorter than any oblique line.

For, draw A C, B C, and D C; then P C is less than P A, P B, or P D. - - *Ch. II, Th. XIX.*

Second: If $AC = BC$, $PA = PB$.

For, the triangles P C A and P C B, having two sides and included angle of the one equal to two sides, and included angle of the other, each to each, are equal. Hence, $PA = PB$.

Third: If $DC > AC$, $PD > PA$.

For, take $CE = AC$, and draw P E.

Then, $PE = PA$, as just proved.

But, $PD > PE$. - - - - - - - - *Ch. II, Th. XIX.*

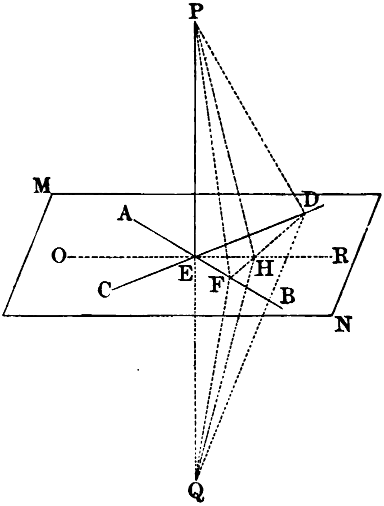
Hence, $PD > PA$. *Q. E. D.*

Cor.—*A perpendicular is the shortest line that can be drawn from a point to a plane.*

Scholium.—The equal oblique lines P A, P B, and P E, meet the plane M N in the circumference of a circle whose centre is C, the foot of the perpendicular, and whose radius is A C.

THEOREM III.

A line which is perpendicular to two lines of a plane at their point of intersection, is perpendicular to the plane.



Let P E be perpendicular to A B and C D at E; then will it be perpendicular to M N.

For, let O R be any other line of the plane M N, passing through E.

Draw D F intersecting A B, C D, and O R.

Produce P E, till E Q = E P.

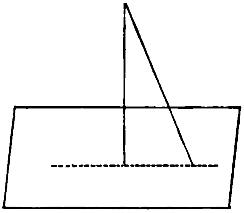
Join P and Q with F, H, and D.

In the triangles P D F and Q D F, P D = Q D, since C D is perpendicular to P Q at its middle point; for a similar reason, P F = Q F; D F is common: hence they are identically equal; and if Q D F be revolved about D F as an axis, Q will coincide with P, and Q H with P H.

Hence, Q H = P H; that is, O R has two of its points, E and H, equally distant from P and Q.

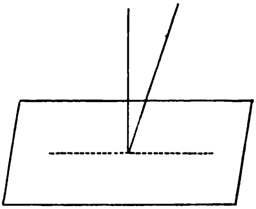
Hence, O R is perpendicular to P Q, or P E is perpendicular to O R; that is, P E is perpendicular to any line of M N; hence, by definition, it is perpendicular to M N. *Q. E. D.*

EXERCISES.



1. Only one perpendicular to a plane can be drawn from a point without the plane.

Suggestion.—Can a triangle have more than one right angle?



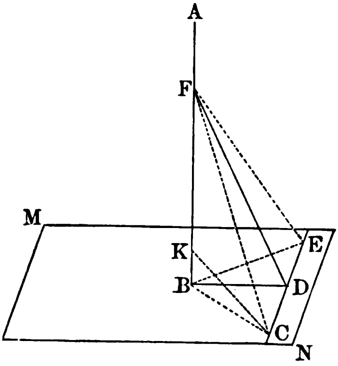
2. Only one perpendicular to a plane can be drawn from a point within the plane.

3. To draw a perpendicular to a plane from a point without a plane *(See Th. II, Sch.)*.

4. To erect a perpendicular to a plane from a point within the plane.

THEOREM IV.

If from the foot of a perpendicular to a plane, a line be drawn at right angles to any line of the plane, and this point of intersection be joined with any point of the perpendicular, the last line will be perpendicular to the line of the plane.



From the foot of the perpendicular A B, let B D be drawn at right angles to C E, any line of the plane M N; and let D be joined with any point of the perpendicular, as F; then will D F be perpendicular to C E.

For take D C = D E.

Join C and E with B and F.

Then B C = B E.

Hence, C F = E F.

Hence, D F has two of its points, D and F, equally distant from C and E.

Hence, D F is perpendicular to C E. *Q. E. D.*

Cor.—*B D measures the shortest distance between A B and C E.*

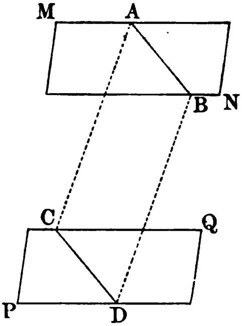
For, let C be any other point in C E than D, and let K be any other point in A B than B; then B D is less than B C; but B C < C K; hence B D < C K.

QUERIES.

1. What kind of triangles are B D C and B D E?
2. B C F and B E F have what two sides and included angle equal, each to each?
3. Where is the right angle in K B C?

THEOREM V.

If a plane intersect two parallel planes, the lines of intersection will be parallel.



Let the plane A D intersect the two parallel planes M N and P Q, in the lines A B and C D; then will A B and C D be parallel.

For, if they are not parallel, they will meet if sufficiently produced, since they lie in the same plane A D.

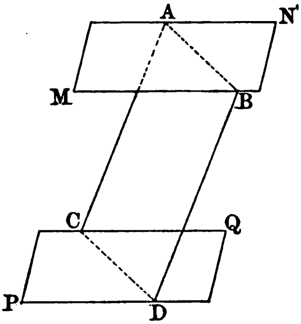
But if the lines meet, the planes M N and P Q, in which they lie, will also meet. But this is contrary to the supposition.

Hence, A B and C D cannot meet; hence, they are parallel.

Q. E. D.

THEOREM VI.

Parallel lines included between parallel planes are equal.



Let A C and B D be any two parallel lines included between the parallel planes M N and P Q; then will A C = B D.

For, conceive a plane to be passed through the parallels; it will intersect M N in A B, and P Q in C D; and these lines of intersection will be parallel - - - - - - - - - *Th. V.*

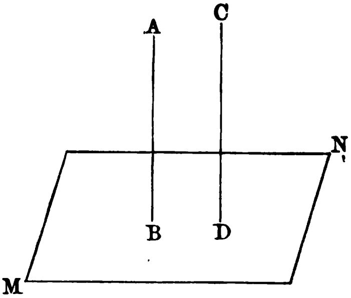
Hence, A D is a parallelogram.

Hence, A C = B D - - - - - - - - *Ch. II, Th. XXIII.*

Q. E. D.

THEOREM VII.

If two straight lines are perpendicular to the same plane, they will be parallel to each other.



For, only one perpendicular to a plane can be drawn from a point within the plane.

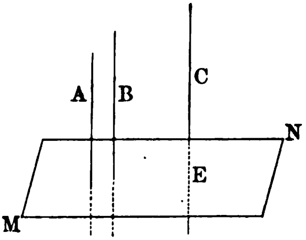
Hence, when a line is perpendicular to a plane, its direction is determined.

A B and C D are both perpendicular to the plane M N; hence, they have the same direction; hence, they are parallel by definition. *Q. E. D.*

Cor. I.—*If one of two parallels is perpendicular to a plane, the other is perpendicular also.*

For, since they are parallel, they have the same direction; hence, if one is at right angles to the plane, the other must be.

Cor. II.—*Two lines parallel to a third in a different plane, are parallel to each other.*

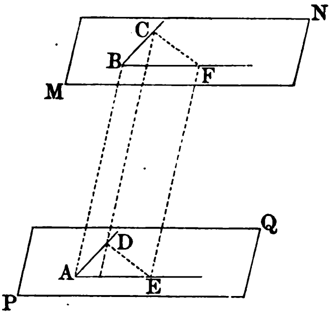


Let A and B be each parallel to C.

Then, if a plane be passed perpendicular to C at E, it will be perpendicular to A and B (*Cor. I*); ∴ A and B, by theorem, are parallel.

THEOREM VIII.

If two angles, not in the same plane, have their sides parallel and in the same direction, the angles will be equal and their planes parallel.



Let the angles A and B lie in the different planes M N and P Q, having their sides parallel and in the same direction; then,

First: A = B.

For, take A D = B C, and A E = B F.

Join A, D, and E with B, C, and F. Draw D E and C F.

Then, A D being equal and parallel to B C, D C is equal and parallel to A B. - - - - - - - - *Ch. II, Th. XVI, XIV.*

A E being equal and parallel to B F, E F is equal and parallel to A B.

Hence, E F and D C are equal and parallel to each other (*Th. VII, Cor. II*).

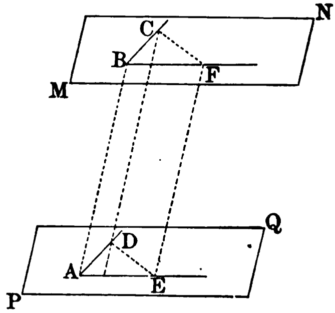
Hence, E C is a parallelogram. - - - - *Ch. II, Th. XVI.*

Hence, E D = C F. - - - - - - - - *Ch. II, Th. XIV.*

Hence, triangles A D E and B C F are equal - *Ch. II, Th. XX.*

Hence, A = B.

Second: M N and P Q are parallel.

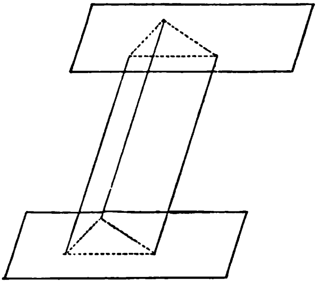


For, conceive a plane to be passed through B F, and let it revolve till it is parallel to P Q.

Since parallel lines included between parallel planes are equal, the revolved plane must cut D C, which is parallel to A B and E F, so that $DC = AB = EF$; hence it will pass through C; hence it will coincide with M N. - - - - - - - - - - - - - - - - - *Th. I.*

Hence, M N and P Q are parallel. *Q. E. D.*

EXERCISES.

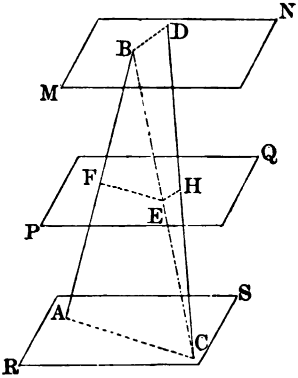


1. If two intersecting planes be cut by parallel planes, the corresponding angles formed by the intersections are equal.

2. If the corresponding extremities of three equal parallel lines, not in the same plane, be joined, the triangles so formed will be equal, and their planes parallel.

THEOREM IX.

If two straight lines are cut by three parallel planes, they will be divided proportionally.



Let the lines A B and C D be cut by the parallel planes M N, P Q, and R S, in the points A, F, and B, C, H, and D; then

$$CH : DH :: AF : BF.$$

For, draw B C, piercing P Q in E. Draw E F and E H, A C and B D.

The plane B C D intersects M N and P Q in B D and E H; hence B D and E H are parallel - *Th. V.*

Hence, C H : D H :: C E : B E - - - - *Ch. VII, Th. I.*

The plane A B C intersects P Q and R S in A C and E F; hence A C and E F are parallel.

Hence, A F : B F :: C E : B E.

∴ C H : D H :: A F : B F. *Q. E. D.*

Cor. I.—*If any number of straight lines be cut by three parallel planes, they will be divided proportionally.*

Cor. II.—*If any number of straight lines be cut by any number of parallel planes, the corresponding segments will be proportional.*

EXERCISES.

1. What position must two lines have, which cannot meet, and yet are not parallel?

2. How, by the use of two carpenter's squares, could you erect a perpendicular to a plane? (*See Th. III.*)

3. How could you determine a point in the floor directly under a given point in the ceiling? (*See Th. II.*)

4. To find a point in a plane equally distant from three given points, not in the plane.

5. What is the locus of all the points in space, which are equally distant from two fixed points?

6. What is the locus of points in space equally distant from three fixed points?

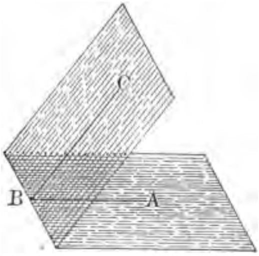
7. What is the locus of points equally distant from two fixed parallel planes?

8. What is the locus of the points in a plane such that the difference of the squares of their distances from two fixed points, not in the plane, shall be constant?

DEFINITIONS.

1. A Diedral[1] Angle is the opening between *two* intersecting planes.

The intersecting planes are the *faces* of the angle, and their line of intersection is the *edge*.



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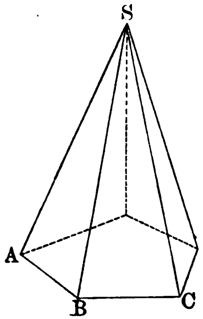
[1] From the Greek, meaning *two-faced.*

2. A diedral angle is *measured* by the plane angle formed by two straight lines, one in each face, perpendicular to the edge at the same point,[1] as A B C.



3. A **Polyedral Angle** is the opening between *three* or more planes proceeding from a common point; or, otherwise, it is the angular space contained by three or more planes, intersecting in but one common point.

The common point is the *vertex;* the lines in which the planes intersect, are the *edges;* the portions of the bounding planes included between the edges, are the *faces,* as S A B and S B C; the plane angles formed by the consecutive edges, are the *facial angles,* as A S B, B S C, *etc.*

4. A **Triedral** Angle is a polyedral angle of three faces; a **Quadriedral**, one of four faces, *etc.*

Three or more planes meeting in one point are said to form at that point a **Solid Angle.**

5. A diedral angle is *right, acute,* or *obtuse,* according as its measure is right, acute, or obtuse.

6. Two planes are perpendicular to each other, or oblique, according as their diedral angle is right or oblique.

THEOREM X.

Any facial angle of a triedral angle is less than the sum of the other two.

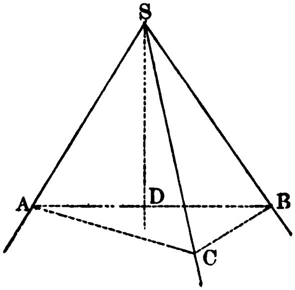
First: If the proposed angle is *less* than either of the other two, it must evidently be less than their *sum.*

Second: If the proposed angle is *equal* to either of the other two, it is still evidently less than their *sum.*

---

[1] Observe, that the two perpendiculars, with respect to the edge, are in *different* planes. An indefinite number of perpendiculars can be drawn to a common point of a given line; as may be illustrated by placing one pencil perpendicular to another, then revolving it around that other as an axis. This exhibits one of the differential features between Plane Geometry and Geometry of Space.

Third: Suppose, finally, that the proposed angle is *greater* than either of the other two, and let A S B be that angle; then will $ASB < ASC + BSC$.



For, in the plane of A S B, make $BSD = BSC$.

Draw A B at pleasure, and take $SC = SD$.

Join C with A and B.

In the triangles B S D and B S C, $SD = SC$, by construction; S B is common; and, by construction,

$$BSD = BSC,$$

∴ (1) $BD = BC$ - - - - - - - *Ch. II, Th. IV.*

But (2) $AB < AC + BC$ - - - *Ch. II, Th. III.*

$(2) - (1) = (3)\ AB - BD < AC\cdot$

that is, $AD < AC$.

Now, in the triangles A S D and A S C, A S is common; $SD = SC$; but $AD < AC$;

∴ $ASD < ASC$ - - *Ch. II, Th. XXVI, Ex. 6.*

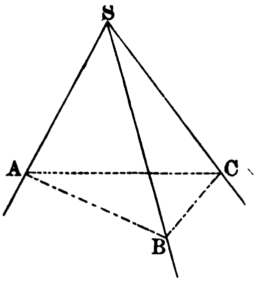
But $BSD = BSC$.

∴ $ASD + BSD < ASC + BSC$;

that is, $ASB < ASC + BSC$. *Q. E. D.*

THEOREM XI.

The sum of the three facial angles of a triedral angle is less than four right angles.



Let S be a triedral angle; then

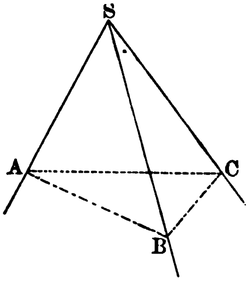
$$ASB + ASC + BSC < 4R.$$

For, join any three points in the edges, forming the triangle A B C.

In the triedral angle A,

$BAS + CAS > BAC$ - *Th. X.*

In the triedral B,



$$ABS + CBS > ABC.$$

In the triedral C,

$$ACS + BCS > ACB.$$

$$\therefore BAS + CAS + ABS + CBS + ACS + BCS > BAC + ABC + ACB.$$

But the first member of this inequation is the sum of the angles at the bases of the triangles, whose common vertex is S, and the second member is the sum of the angles of the triangle A B C, which is 2 R.

Hence, denoting the first member by S′, we have

$$(1) \qquad S' > 2\,R.$$

Now, the sum of *all* the angles of the triangles, whose common vertex is S, is 6 R.

Hence, denoting the sum of the facial angles by S″, we have

$$(2) \quad S' + S'' = 6\,R$$

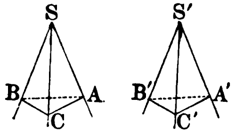
$$\text{But,} \qquad S' > 2\,R.$$

$$(2) - (1) = (3) \qquad S'' < 4\,R. \qquad \textit{Q. E. D.}$$

EXERCISE.

If the plane angles formed by the edges of two triedral angles are equal, each to each, the planes of the equal angles are equally inclined to each other.

Suggestion.—At any point of S C, as C, draw the perpendiculars A C, B C. A C B measures inclination of faces A S C, B S C. Take S′ C′ = S C. Draw perpendiculars A′ C′, B′ C′. Prove right triangles B S C, B′ S′ C′, equal; also A S C = A′ S′ C′, and A S B = A′ S′ B′. Then A B C = A′ B′ C′. Hence angle A C B = angle A′ C′ B′. Hence inclination of planes A S C, B S C, equals that of A′ S′ C′, B′ S′ C′. Proceed in like manner with the planes of the other equal angles.

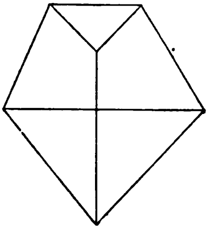


Scholium.—The triedral angles are congruently equal, if the planes of the equal plane angles are similarly placed, since they are superposable; they are equal by symmetry, if the planes are not similarly placed.

CHAPTER XIII.

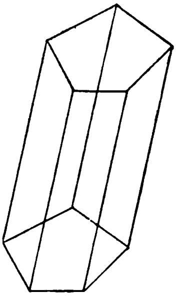
THE PRISM.

DEFINITIONS.



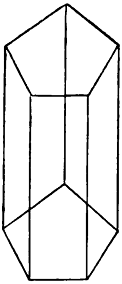
1. A **Polyedron** is a volume, or portion of space, bounded by polygons.

The bounding polygons are called *faces;* the lines in which the faces meet are called *edges;* the points in which the edges meet are called *vertices.*

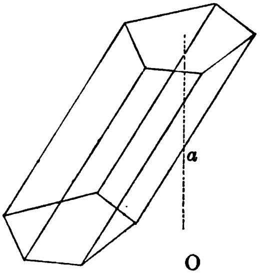


2. A **Prism** is a polyedron, two of whose faces are equal polygons, having their homologous sides parallel; while their other faces are parallelograms.

The equal parallel polygons are called *bases,* one the *upper* and the other the *lower* base; the parallelograms together make up the *lateral* or *convex surface;* the lines in which the parallelograms meet are called *lateral edges.*

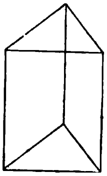


3. A **Right Prism** is one whose lateral edges are perpendicular to its bases.

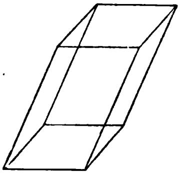


4. An **Oblique Prism** is one whose lateral edges are oblique to its bases.

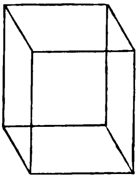
5. The **Altitude** of a prism is the perpendicular distance between the planes of its bases.



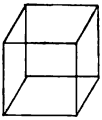
6. A **Triangular** prism is one whose bases are triangles; a **Quadrangular** prism, one whose bases are quadrilaterals, etc.



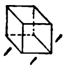
7. A **Parallelopiped** is a prism whose bases are parallelograms. Consequently, it is a polyedron, all of whose faces are parallelograms.



8. A **Rectangular** parallelopiped is one whose faces are all rectangles. Hence, it is a right prism.



9. A **Cube** is a rectangular parallelopiped, all of whose faces are squares.



10. The **Unit of Volume** is a cube whose edge is the linear unit.

Cor. I.—*Any lateral edge of a right prism is equal to the altitude* (*See Def. 3*).

Cor. II.—*Any lateral edge of an oblique prism is greater than the altitude.*

Note.—A cube is the unit of measure for all solids and fluids. For instance, when a cord of wood is said to be 128 cubic feet, the meaning is, that the material in a cord is sufficient to make a pile as large as 128 boxes (*cubes*), whose faces are each a square foot, and whose edges are the linear foot.

By a gallon of water (231 cubic inches), is meant water enough to fill 231 small cubes, whose faces are square inches, and whose edges are the linear inch.

QUERY.

In speaking of a yard of thread, a yard of cloth, and a yard of earth, in what different senses is the word *yard* employed?

MEASURES OF VOLUMES.

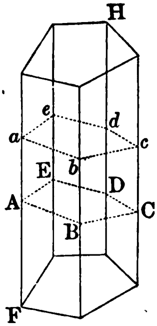
1728 Cubic Inches = 1 Cubic Foot.
27 “ Feet = 1 “ Yard.
4492⅛ “ Feet = 1 “ Rod.
231 “ Inches = 1 Liquid Gallon.
8316 “ Inches = 1 Barrel.
2150.42 “ Inches = 1 Bushel.
128 “ Feet = 1 Cord.

THEOREM I.

Parallel sections of a prism are equal polygons.

Let the prism F H be cut by parallel planes, forming the sections A D and *a d;* then will

A D = *a d.*



For all the sides of A D are parallel to the corresponding sides of *a d* - - - *Ch. IX, Th. V.*

Hence, the two polygons are mutually equiangular. - - - - - - - - *Ch. XII, Th. VIII.*

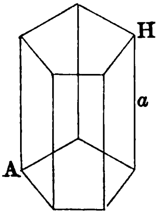
Also, the sides of A D are equal to the corresponding sides of *a d*, since parallel lines included between parallels are equal.

Hence, A D and *a d* are identically equal. *Q. E. D.*

Cor.—*The bases of a prism are equal to any parallel section.*

THEOREM II.

The lateral surface of a right prism is equal to the perimeter of the base multiplied by the altitude.



Let A H be a right prism, S its lateral surface, p the perimeter of its base, and a its altitude; then

$$S = p \times a.$$

For, the lateral surface is equal to the sum of the rectangles which compose it.

The altitude of each rectangle = altitude of the prism = a.

The bases of the rectangles together make the perimeter of the base of the prism.

And the area of any rectangle = its base multiplied by its altitude.

Hence, the sum of the rectangles $= p \times a$. Hence,

$$S = p \times a.$$ *Q. E. D.*

EXERCISES.

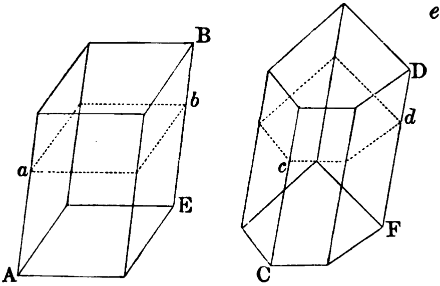
1. The lateral surfaces of two right prisms are proportional to the products of their perimeters by their altitudes.

2. If two right prisms have equal altitudes, their lateral surfaces are proportional to the perimeters of their bases.

3. The perimeters of their bases being equal, their lateral surfaces are proportional to their altitudes.

THEOREM III.

Any two prisms having equal bases and equal altitudes, are equal.



Let A B and C D be two prisms having equal bases and equal altitudes; then

$$A B = C D.$$

For, place their lower bases in the same plane;

then by a cutting plane parallel to the plane of the bases, conceive the two prisms to be divided into indefinitely thin plates, as $a\,b$ and $c\,d$, piled one upon the other.

Each prism will be composed of the same number of plates, since each has the same altitude.

The plates, being indefinitely thin, may, without error, be regarded as plane figures; and any plate, as $a\,b$, is equal to its corresponding plate $c\,d$.

For, $a\,b = \mathrm{A\,E}$,

and $c\,d = \mathrm{C\,F}$. - - - - - - - *Th. I, Cor.*

But, $\mathrm{A\,E} = \mathrm{C\,F}$, by supposition.

$\therefore\ a\,b = c\,d$.

Hence, the sum of the plates composing A B is equal to the sum of the plates composing C D.

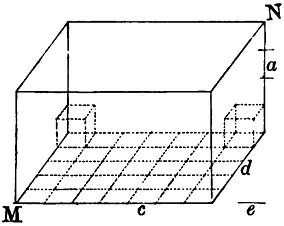
$\therefore\ \mathrm{A\,B} = \mathrm{C\,D}$. *Q. E. D.*

Cor. I.—*Any prism is equal to a rectangular parallelopiped having an equal base and the same altitude.*

Cor. II.—*Any parallelopiped is equal to a rectangular parallelopiped having an equal base and the same altitude.*

THEOREM IV.

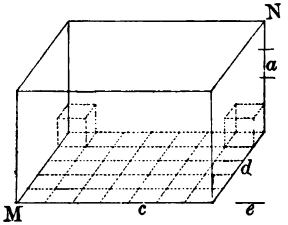
The volume of a rectangular parallelopiped is equal to the product of its base by its altitude.



Let M N be a rectangular parallelopiped whose edges are c, d and a; then will its volume be equal to the product of its base $(c \times d)$ by its altitude, a.

For, let the linear unit e be the common measure of the three edges a, d, and c; and suppose it to be contained in a 4 times, in d 5 times, and in c 7 times.

Through the points of division, let planes be passed parallel to the faces of the prism.



Then the intersections of these planes will each be equal to the linear unit, and the angles which they form will be right angles.

Hence, the parallelopiped will be divided into equal cubes, equal because they have equal bases and equal altitudes (*Th. III*).

These little cubes are arranged in layers; and the number of cubes in one layer, for example, the first, is evidently equal to the number of *squares* in the base, that is, the *number* of superficial units in the base, which is

$c \times d$ - - - - *Ch. VI, Th. V.*

The *whole* number of cubes equals the number in one layer multiplied by the number of layers.

But the number of layers equals the number of linear units in the altitude.

Hence, the whole number of cubes equals $(c \times d)\ a$.

Hence, the volume of M N

$= (c \times d)\ a = c \times d \times a.$ *Q. E. D.*

Scholium.—This is equally true when the edges are incommensurable.

For, if the linear unit be taken indefinitely small, it may be applied to each of the edges with an indefinitely small remainder; and the prism, *so far*, is, by the preceding case, measured by the product of its base by its altitude.

The unmeasured part of both the edges and prism grows still less as the linear unit becomes less. But at each successive step, *indefinitely*, the prism, *so far as its edges are measured*, is equal to its base multiplied by its altitude; and this, being true up to the limit, is true at the limit - *Ch. X, Th VI, Sch.*

Cor. I.—*The volume of a rectangular parallelopiped is equal to the product of its three dimensions,—length, breadth, and altitude.*

Cor. II.—*The volume of a cube is the third power of its edge.*

For, in this case, the length, breadth, and altitude are equal.

Cor. III.—*The volume of any parallelopiped is equal to the product of its base by its altitude.*

Let V denote the volume of any parallelopiped, B its base, and a its altitude.

Let V′ denote the volume of a rectangular parallelopiped having an equal base B′, and an equal altitude a'.

Then $V' = B' \times a'$.

But $B' = B$, and $a' = a$.

$\therefore B' \times a' = B \times a$.

$\therefore V' = B \times a$.

But $V = V'$ - - - - *Th. III, Cor. II.*

$\therefore V = B \times a$.

THEOREM V.

The volume of any triangular prism is equal to the product of its base by its altitude.

Let V denote the volume of any triangular prism, T its base, and a its altitude; then will

$$V = T \times a.$$

For, let V′ denote the volume of a rectangular parallelopiped having an equal altitude a', and an equal base P.

Then $V' = P \times a'$ - - - - - - - *Th. IV.*

But $T = P$,

and $a = a'$;

$\therefore T \times a = P \times a'$.

$\therefore V' = T \times a$.

But $V = V'$ - - - - - - *Th. III, Cor. I.*

$\therefore V = T \times a$

= its base multiplied by its altitude.

Q. E. D.

THEOREM VI.

The volume of any prism is equal to the product of its base by its altitude.

Let V denote the volume of any prism, B its base, and a its altitude; then

$$V = B \times a.$$

For, let V′ denote the volume of a triangular prism having an equal altitude a', and an equal base T.

$$\begin{array}{lll}
\text{Then} & V' = T \times a' \text{ - - - - - - -} & \textit{Th. V.} \\
\text{But,} & B \times a = T \times a'. & \\
\therefore & V' = B \times a. & \\
\text{But,} & V = V' \text{ - - - - - - - -} & \textit{Th. III.} \\
\therefore & V = B \times a. & \textit{Q. E. D.}
\end{array}$$

EXERCISES.

1. Any two prisms are proportional to the products of their bases by their altitudes.

2. Any two prisms having equal altitudes, are proportional to their bases.

3. Any two prisms having equal bases, are proportional to their altitudes.

Note.—It is worthy of notice, that the volume of a prism depends conjointly on its altitude, and the area of its base;—with the same base, the volume will vary as the altitude; with the same altitude, the volume will vary as the base.

CHAPTER XIV.

THE PYRAMID.

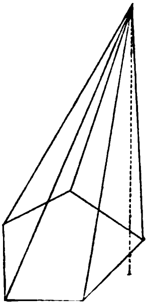
DEFINITIONS.

1. A **Pyramid** is a polyedron, or volume, having a polygon for its base, and triangles for its lateral faces.

The *vertex* of the pyramid is the common vertex of the triangles; the *lateral surface* of the pyramid is the sum of its lateral faces; the *lateral edges* of the pyramid are the lines in which the lateral faces meet.

2. The **Altitude** of a pyramid is the perpendicular distance from its vertex to the plane of its base.

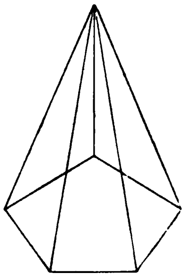
3. A **Triangular** pyramid is one whose base is a triangle.



4. A **Right Pyramid** is one whose base is a regular polygon, and whose perpendicular from the vertex pierces the centre of the base.

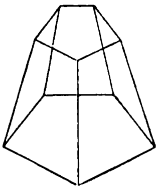
5. The **Slant Height** of a right pyramid is the altitude of any lateral face.

6. A **Frustum** of a pyramid is the portion of a pyramid included between the base and a parallel section.



If the cutting plane is not parallel to the base, the intercepted portion is called a *Truncated* pyramid.

7. The **Altitude** of a **Frustum** is the perpendicular distance between the planes of its bases.

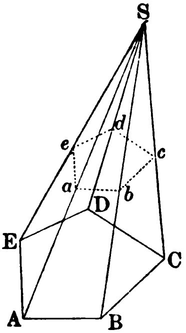


8. The **Slant Height** of a **Frustum** of a right pyramid is the altitude of any lateral face.

The lateral faces of such a frustum, as appears from the diagram, are *trapezoids*.

THEOREM I.

Any section of a pyramid parallel to the base is similar to the base.



In the pyramid $S-ABCDE$, let the section $abcde$ be parallel to the base; then will it be similar to the base.

For, all of its sides are parallel to the corresponding sides of the base. - *Ch. XII, Th. V.*

Hence, the base and the section are mutually equiangular. - - - - -*Ch. XII, Th. VIII.*

Since ab and bc are parallel to AB and BC, the triangles Sab and SAB are similar (*Ch. VIII, Th. I*); also Sbc and SBC.

Hence, $Sb : SB :: ab : AB$,

and $Sb : SB :: bc : BC$.

$\therefore ab : AB :: bc : BC$.

Similarly, $bc : BC :: cd : CD$, *etc.*

$\therefore ab : AB :: bc : BC :: cd : CD$, *etc.*

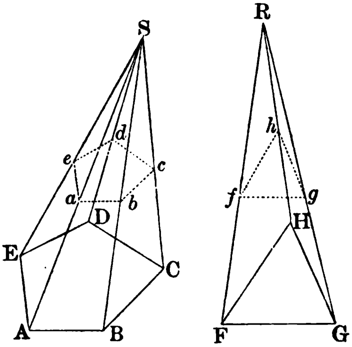
Hence, the section and the base have their homologous sides proportional.

Hence, they are similar. *Q. E. D.*

THEOREM II.

If two pyramids have the same altitude, sections parallel to the bases and equally distant from them, are proportional to the bases.

Let $S-ABCDE$ and $R-FGH$ be two pyramids, whose bases are in the same plane, and in which $abcde$ and fgh are



sections parallel to the bases and equally distant from them; then, the altitudes being equal,

$$ad : fgh :: AD : FGH.$$

For, since AD and ad are similar (*Th. I*),

$$AD : ad :: \overline{AB}^2 : \overline{ab}^2.$$

But, $AB : ab :: SA : Sa$, whence,

$$\overline{AB}^2 : \overline{ab}^2 :: \overline{SA}^2 : \overline{Sa}^2.$$

$$\therefore AD : ad :: \overline{SA}^2 : \overline{Sa}^2.$$

Similarly, $FGH : fgh :: \overline{RF}^2 : \overline{Rf}^2$.

But, $SA : Sa :: RF : Rf$ - *Ch. XII, Th. IX.*

$\therefore$ $\overline{SA}^2 : \overline{Sa}^2 :: \overline{RF}^2 : \overline{Rf}^2$.

$\therefore$ $AD : ad :: FGH : fgh$, or

$ad : fgh :: AD : FGH$. *Q. E. D.*

Cor.—*If two pyramids, having equal altitudes, have equal bases, any two sections equally distant from the bases are equal.*

For, from the last proportion, if

$AD = FGH$, then $ad = fgh$.

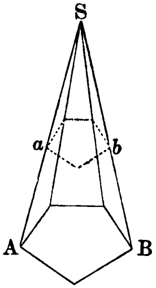
Scholium.—The student will notice that SA and RF are two lines cut by three parallel planes, the first plane being the plane of the bases, and the third passing through the vertices S and R.

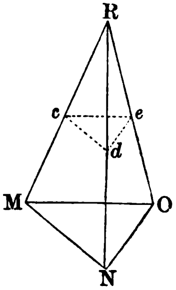
On the same principle, the altitude and any edge are divided proportionally at these points.

THEOREM III.

Any two pyramids having equal bases and equal altitudes, are equal.

Let $S - AB$ and $R - MNO$ be two pyramids having equal bases and equal altitudes; then will they be equal.





For, conceive the two pyramids to be divided into indefinitely thin plates, by planes parallel to the plane of the bases.

Each pyramid will be divided into the same number of plates, since they have the same altitude.

And any plate, as *a b*, of the one is equal to the corresponding plate, *c d e*, of the other (*Th. II, Cor.*).

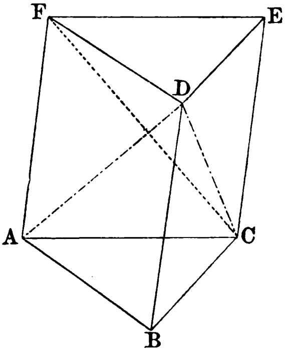
Hence, the sum of the plates composing one is equal to the sum of the plates composing the other; that is,

S – A B = R – M N O. *Q. E. D.*

Cor.—*Any pyramid is equal to a triangular pyramid having an equal base and the same altitude.*

THEOREM IV.

Any triangular prism may be divided into three equal triangular pyramids.



Let A B C—E be a triangular prism; then may it be divided into three equal triangular pyramids.

For, pass a plane through A C and the vertex D, cutting off the pyramid A B C – D.

Pass a plane through D F and the vertex C, cutting off the pyramid C – D E F.

Then there will remain a third pyramid A C F – D.

Now, A B C – D = C – D E F. - - - - - - - *Th. III.*

But C — D E F is identical with C E F — D, D being regarded as the vertex, and C E F as the base; and A C F — D = C E F — D, because their bases are halves of the same parallelogram, and since they have a common altitude, the perpendicular from D to the plane A C E F. Hence,

$$ACF - D = ABC - D = C - DEF. \qquad \textit{Q. E. D.}$$

Cor. I.—*A triangular pyramid is one-third of the prism having an equal base and the same altitude.*

Cor. II.—*The volume of a triangular pyramid is one-third of the product of its base by its altitude.*

For, let V denote the volume of a triangular pyramid, T its base, and a its altitude.

Let V′ denote the volume of a prism having an equal base B, and an equal altitude a'.

Then, $V' = B \times a'$.

But, $V = \frac{1}{3} V' = \frac{1}{3} B \times a'$.

But, $B \times a' = T \times a$, whence

$\frac{1}{3} B \times a' = \frac{1}{3} T \times a$.

Hence, $V = \frac{1}{3} T \times a$.

THEOREM V.

The volume of any pyramid is equal to one-third of the product of its base by its altitude.

Let V denote the volume of any pyramid, B its base, and a its altitude; then

$$V = \tfrac{1}{3} B \times a.$$

For, let V′ denote the volume of a triangular pyramid having an equal base T, and an equal altitude a'.

Then $V' = \frac{1}{3} T \times a'$ - - - *Th. IV, Cor. II.*

But $V = V'$ - - - - - - - *Th. III, Cor.*

∴ $V = \frac{1}{3} T \times a'$.

But $\frac{1}{3} T \times a' = \frac{1}{3} B \times a$ - - - - - - - - - (?)

∴ $V = \frac{1}{3} B \times a$. *Q. E. D.*

Cor.—*The volume of any pyramid is one-third of the prism having an equal base and the same altitude.*

EXERCISES.

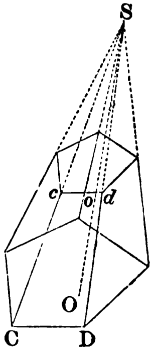
1. Any two pyramids are proportional to the products of their bases by their altitudes.

2. Any two pyramids having equal altitudes, are proportional to their bases.

3. Any two pyramids having equal bases, are proportional to their altitudes.

THEOREM VI.

The volume of a frustum of any pyramid is equal to one-third of the product of its altitude by the sum of its bases and a mean proportional between them.



Let F denote the frustum, a its altitude, B its lower base, and b its upper base; then

$$F = \tfrac{1}{3}a\,(B + b + \sqrt{B \times b}).$$

For, let P denote the entire pyramid, and h its altitude; let p denote the portion above the frustum, then its altitude will be $h - a$.

Now, $F = P - p$.

But $P = \tfrac{1}{3}B \times h$,

and $p = \tfrac{1}{3}b\,(h - a)$;

$$\therefore (1)\ F = \tfrac{1}{3}B \times h - \tfrac{1}{3}b\,(h - a)$$
$$= \tfrac{1}{3}B \times h - \tfrac{1}{3}b \times h + \tfrac{1}{3}b \times a$$
$$= \tfrac{1}{3}\,[h\,(B - b) + b \times a].$$

Again, $B : b :: \overline{CD}^2 : \overline{cd}^2$ - - - - - - (?)

But, $\overline{CD}^2 : \overline{cd}^2 :: \overline{SD}^2 : \overline{Sd}^2$ - - - - - - (?)

$\therefore$ $B : b :: \overline{SD}^2 : \overline{Sd}^2$ - - - - - - (?)

But, $\overline{SD}^2 : \overline{Sd}^2 :: \overline{SO}^2 : \overline{So}^2$ - - - - - - (?)

$\therefore$ $B : b :: h^2 : (h - a)^2$ - - - - (?)

$$\therefore \quad \sqrt{B} : \sqrt{b} :: h : h - a \quad - - - - - - \quad (?)$$

$$\therefore \sqrt{B}\,(h - a) = h\sqrt{b} \quad - - - - - - - - - \quad (?)$$

Or, $h\sqrt{B} - a\sqrt{B} = h\sqrt{b};$

$\therefore \quad h\sqrt{B} - h\sqrt{b} = a\sqrt{B};$

$\therefore \quad (2)\ h(\sqrt{B} - \sqrt{b}) = a\sqrt{B}.$

$$(2) \times (\sqrt{B} + \sqrt{b}) =$$

$$(3) \quad h(\sqrt{B} - \sqrt{b})(\sqrt{B} + \sqrt{b}) = a\sqrt{B}(\sqrt{B} + \sqrt{b})$$

$$= a \times B + a\sqrt{B \times b}.$$

And $h(\sqrt{B} - \sqrt{b})(\sqrt{B} + \sqrt{b}) = h(B - b);$

$\therefore \quad (4) \quad h(B - b) = a \times B + a\sqrt{B \times b}.$

Substituting (4) in (1),

$$F = \tfrac{1}{3}(a \times B + a\sqrt{B \times b} + b \times a)$$

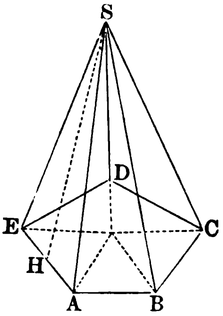
$$= \tfrac{1}{3}a(B + \sqrt{B \times b} + b)$$

$$= \tfrac{1}{3}a(B + b + \sqrt{B \times b})$$

= *one-third of the product of the altitude by the sum of the bases and a mean proportional between them.* *Q. E. D.*

THEOREM VII.

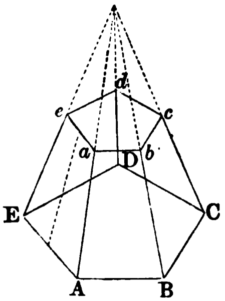
The lateral surface of a right pyramid is equal to the perimeter of its base multiplied by one-half its slant height.



[The student may give the proof. By definition, the perpendicular from S pierces the centre of the base, and this centre is equi-distant from all the vertices; hence the edges are equal (*Ch. IX, Th. II*); hence the lateral faces are equal isosceles triangles (*Ch. II, Th. XVII*). Find the area of each triangle. The altitude of each equals slant height.]

THEOREM VIII.

The lateral surface of a frustum of a right pyramid is equal to one half the sum of the perimeters of its bases, multiplied by the slant height.

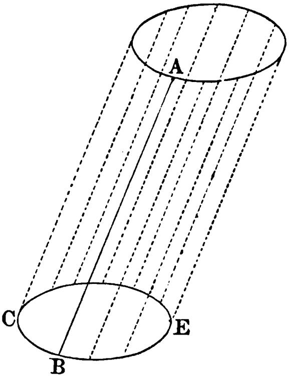


[The proof is obvious. The area of a trapezoid equals what? Find each lateral face, and add.]

CHAPTER XV.

THE CYLINDER.

DEFINITIONS.



1. A **Cylindrical Surface** is a curved surface generated by the movement of a straight line which continually touches a given curve, and remains constantly parallel to its first position.

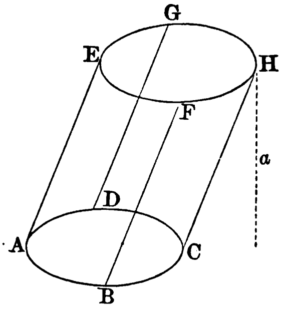
Thus, the line A B, moving so as continually to touch the given curve C E, in parallel positions, generates a cylindrical surface.

2. The moving line is the **Generatrix.**

3. The curve within which the generatrix moves, is the **Directrix.**[1]

4. A line of the surface corresponding to any position of the generatrix is an **Element.**

[Evidently, if the generatrix is of indefinite length, a cylindrical surface of indefinite extent is generated. The directrix may be any curve whatever—closed or open. The possible cases are in this treatise limited to one, that in which the directrix is a *circle.*]



5. A **Cylinder** is a volume bounded by a cylindrical surface and two parallel planes, as A H.

6. The **Convex Surface** of a cylinder is its cylindrical surface.

7. The **Bases** of a cylinder are its plane surfaces, as A B C D and E F H G.

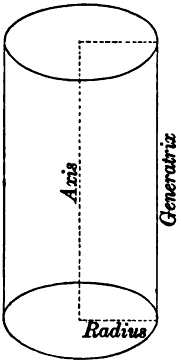
---

[1] The curve is conceived as *directing* the motion of the generatrix.

[The term *cylinder*, then, is assumed to denote one whose bases are *circles*.]

8. The **Altitude** of a cylinder is the perpendicular distance from one base to the plane of the other, as *a*.

9. The **Radius** of a cylinder is the radius of the base.



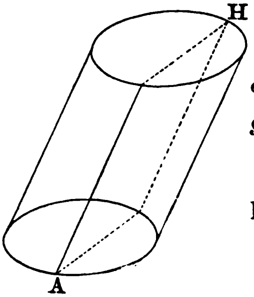
10. A **Right Cylinder**[1] is one whose elements are perpendicular to its bases.

We may conceive a right cylinder to be generated by the revolution of a rectangle about one of its sides as an axis.

11. An **Oblique Cylinder** is one whose elements are oblique to its bases.

12. **Similar Right Cylinders** are those which are generated by similar rectangles revolving about homologous sides.

13. A prism is **Inscribed** in a cylinder when the bases of the prism are inscribed in those of the cylinder, the edges of the prism coinciding with elements of the cylinder.



Cor. I.—*Every section of a cylinder embracing two elements, is a parallelogram.*

For these elements are, by definition, parallel; and they are equal (*Ch. XII, Th. VI*); hence A H is a parallelogram.

Cor. II.—*The bases of a cylinder are equal.*

For every position of the generatrix is parallel to its first one.

Cor. III.—*Any two parallel sections of a cylinder are equal.*

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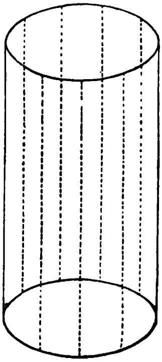
[1] Right cylinder with circular base, sometimes called 'Cylinder of Revolution.'

Cor. IV.—*All sections of a cylinder parallel to its bases, are equal circles.*



Cor. V.—*The altitude of an oblique cylinder is less than any element.*

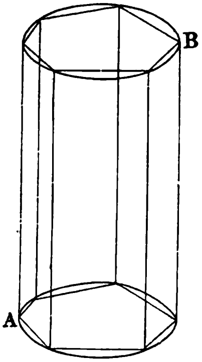
For the perpendicular is the shortest distance from one base to the plane of the other.

Cor. VI.—*The altitude of a right cylinder is equal to any element.*

For the elements are all perpendicular to the planes of the bases, which are parallel.

THEOREM I.

The convex surface of a right cylinder is equal to the circumference of its base multiplied by its altitude.



Let A B be a right cylinder; then will its convex surface be equal to the circumference of its base multiplied by its altitude.

For, inscribe within the cylinder a prism whose bases are regular polygons.

Then the lateral surface of this prism equals the perimeter of its base multiplied by its altitude, whatever be the number of sides or faces. - - - - - *Ch. XIII, Th. II.*

But when the number of sides becomes infinite (*Ch. X, Th. VI, Sch.*), the prism coincides, in every respect, with the cylinder.

Hence, the convex surface of the cylinder equals the circumference of its base multiplied by its altitude. *Q. E. D.*

Cor. I.—*Let* S *denote the lateral surface,* C *the circumference of the base, and* a *the altitude; then*

$S = C \times a = 2\pi r \times a$ - - - - *Ch. X, Th. XII.*

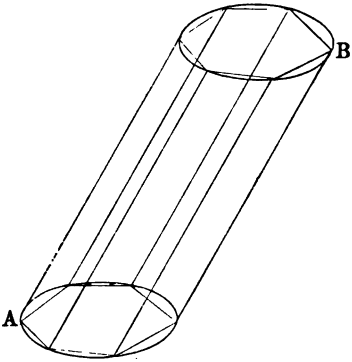
Cor. II.—*Let* S′ *denote the entire area* (*including the areas of the bases*); *then*

$$\begin{aligned} S' &= S + \pi r^2 + \pi r^2 \\ &= 2\pi r \times a + 2\pi r^2 \\ &= 2\pi r(a + r). \quad - \; - \; - \quad \textit{Ch. X, Th. XI.} \end{aligned}$$

Scholium.—The truth of this theorem is clearly evident from the consideration, that if the cylindrical surface were cut open and spread out, it would assume the form of a rectangle, whose base and altitude would be the circumference and altitude of the cylinder.

THEOREM II.

The volume of a cylinder is equal to the product of its base by its altitude.



Let A B be a cylinder; then will its volume be equal to the product of its base by its altitude.

For, inscribe within the cylinder a prism whose bases are regular polygons.

Then the volume of the prism equals the product of its base by its altitude, whatever be the number of sides.

But when the number of sides is infinite, the prism concides, in every respect, with the cylinder.

Hence, the volume of the cylinder equals the product of its base by its altitude. *Q. E. D.*

Cor.—*Let* V *denote the volume,* r *the radius of the base, and* a *the altitude; then* $V = \pi r^2 \times a = \pi r^2 a$.

EXERCISES.

1. The convex surfaces of cylinders having equal altitudes, are proportional to the radii of their bases.

2. Cylinders having equal bases are proportional to their altitudes.

3. Similar right cylinders, being those which are generated by similar rectangles revolving about homologous sides, prove that the convex surfaces of two similar right cylinders are proportional to the squares of their radii or to the squares of their altitudes.

4. Two similar right cylinders are proportional to the cubes of their radii or to the cubes of their altitudes.

Suggestion.— $\dfrac{V}{V'} = \dfrac{\pi\, r^2\, a}{\pi\, r'^2\, a'}$ - - - - - - - - - - - - - - - - (?)

$\dfrac{a}{a'} = \dfrac{r}{r'}$. - - - - - - - - - - - - - - - - - (?)

5. What is the convex surface of a cylinder whose altitude is 16 feet, and the radius of whose base is 3 feet?

6. The convex surface of a cylinder is 3141.6, and its altitude 50? what is the diameter of its base?

7. What is the entire surface of a cylinder whose altitude is 16 feet, and diameter of the base 2 feet?

8. What is the volume of a cylinder whose altitude is 18 feet, and the circumference of whose base is 6½ feet?

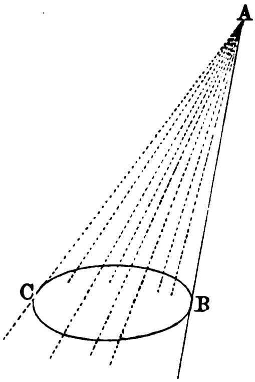
9. Required the cubic contents of a cylindrical log whose length is 16 feet, and the circumference of whose ends is 10 feet.

10. The volume of a cylinder is 318.32 cubic feet, and the circumference of its base is 20 feet; required its altitude.

CHAPTER XVI.

THE CONE.

DEFINITIONS.

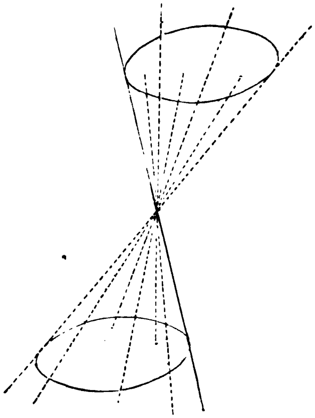


1. A **Conical Surface** is a curved surface generated by the movement of a straight line which passes through a fixed point and continually touches a given curve.

Thus, the line A B, passing through the fixed point A, and moving so as continually to touch the curve B C, generates a conical surface.

2. The moving line is the **Generatrix.**

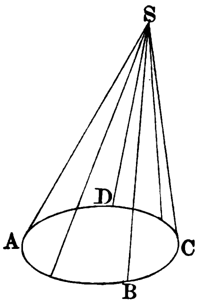
3. The curve which the generatrix continually touches, is the **Directrix.**



4. A line of the surface corresponding to any position of the generatrix, is an **Element.**

[Evidently, if the generatrix extends indefinitely in both directions, the conical surface will consist of two portions, on opposite sides of the fixed point, an upper and a lower, which are called **Nappes.**

The directrix may be any curve whatever, closed or open. As in the case of the cylinder, the possible cases are here limited to one, that in which the directrix is a *circle.*]



5. A **Cone** is a volume bounded by a conical surface and a plane surface.

6. The **Convex Surface** of a cone is its conical surface.

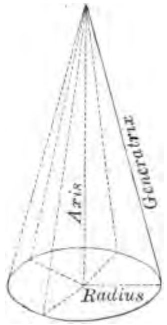
7. The **Base** of a cone is its plane surface, as A B C D.

[The term *cone*, then, is assumed to denote one whose base is a circle.]

8. The **Vertex** of a cone is the fixed point through which all the elements pass, as S.

9. The **Axis** of a cone is the straight line drawn from its vertex to the centre of its base.

10. The **Altitude** of a cone is the perpendicular distance from the vertex to the plane of the base.



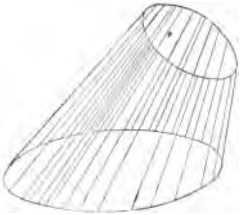
11. A **Right Cone**[1] is one in which the axis is perpendicular to the base.

We may conceive a right cone to be generated by the revolution of a right triangle about one of its perpendicular sides.

12. The **Slant Height** of a right cone is any position of the generatrix—the hypotenuse of the revolving triangle.

13. **Similar Right Cones** are those which are generated by similar right triangles revolving about homologous sides.

14. A **Truncated Cone** is the portion of a cone included between the base and a section cutting the convex surface.



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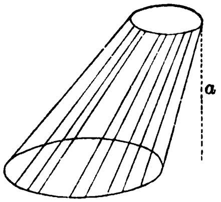
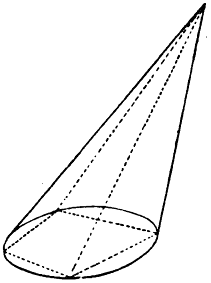
[1] Right *circular* cone; sometimes called 'Cone of Revolution.'



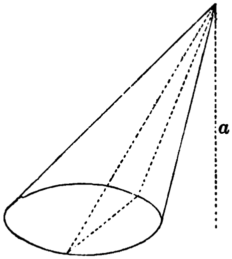
15. If the cutting section is parallel to the base, the truncated cone is called a **Frustum** of a cone.

16. The **Slant Height** *of a Frustum of a Right Cone* is the portion of an element included between its bases.



17. A pyramid is **Inscribed** in a cone when its base is inscribed in that of the cone, and its vertex coincides with that of the cone.

18. A frustum of a pyramid is inscribed in a frustum of a cone when its bases are inscribed in those of the frustum of the cone.



Cor. I.—*Any section of a cone passing through its vertex, is a triangle.*

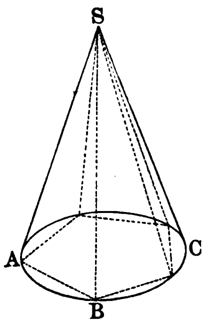
Cor. II.—*Any section of a cone parallel to its base, is a circle.*

Cor. III.—*The altitude of an oblique cone is less than its axis.*

Cor. IV.—*The altitude of a right cone is equal to its axis.*

THEOREM I.

The convex surface of a right cone is equal to the circumference of its base multiplied by one-half its slant height.



Let S — A B C be a right cone; then will its lateral surface equal the circumference of its base multiplied by one-half its slant height.

For, inscribe within it a right pyramid. Then the lateral surface of this pyramid equals the perimeter of its base multiplied by one-half its slant height, whatever be the number of sides of its base (*Ch. XIV, Th. VII*).

But when the number of sides is indefinitely great, or infinite, the pyramid coincides in every respect with the cone.

Hence, the convex surface of the cone equals the circumference of its base multiplied by one-half its slant height.

Q. E. D.

Cor. I.—*If* S *denotes the convex surface of the cone,* C *the circumference of the base,* $\boldsymbol{r}$ *the radius of the base, and* $\boldsymbol{h}$ *the slant height; then, as the algebraic expression of the theorem,*

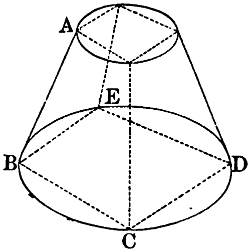
$$S = C\tfrac{1}{2}\boldsymbol{h} = 2\pi\boldsymbol{r} \times \tfrac{1}{2}\boldsymbol{h} = \pi\boldsymbol{r}\boldsymbol{h}.$$

Cor. II.—*The area of the base being* $\pi\boldsymbol{r}^2$, *the entire surface of the cone* $= \pi\boldsymbol{r}\boldsymbol{h} + \pi\boldsymbol{r}^2 = \pi\boldsymbol{r}(\boldsymbol{h} + \boldsymbol{r})$.

Scholium.—The truth of this theorem is evident from the consideration, that if the cone were unfolded, its surface would form a sector of a circle whose arc would be the circumference of its base, and whose radius would be any of the elements—all the elements of a right cone being equal.

THEOREM II.

The convex surface of a frustum of a right cone is equal to the sum of the circumferences of its bases, multiplied by one-half the slant height.



Let A – B C D E be a frustum of a right cone; then will its convex surface be equal to the sum of the circumferences of its bases, multiplied by half the slant height.

For, inscribe within it a frustum of a right pyramid.

Then the lateral surface of this frustum equals the sum of the perimeters of its bases, multiplied by half the slant height, whatever be the number of its faces.

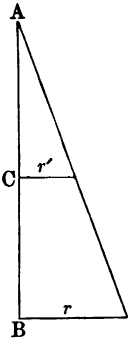
But when the number of faces becomes infinite, the inscribed frustum coincides in every respect with the frustum of the cone.

Hence, the frustum of a right cone equals the sum of the circumferences of its bases, multiplied by half its slant height.

Q. E. D.

Cor.—*If* $\boldsymbol{r}$ *denotes the radius of the lower base,* $\boldsymbol{r'}$ *the radius of the upper, and* $\boldsymbol{h}$ *the slant height, then, as the algebraic expression of the theorem, convex surface* $= \frac{1}{2}\,\boldsymbol{h}\,(2\,\pi\,\boldsymbol{r} + 2\,\pi\,\boldsymbol{r'}) = \pi\,\boldsymbol{h}\,(\boldsymbol{r} + \boldsymbol{r'})$.

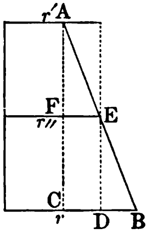
EXERCISES.



1. The convex surface of a right cone is equal to the product of its slant height by the circumference of the circle parallel to the base, and midway between the base and the vertex.

Suggestion.—How does the radius of the midway circle compare with the radius of the base?

$$AC : AB :: r' : r.$$



2. The convex surface of a frustum of a right cone is equal to the product of its slant height by the circumference of the circle midway between its bases.

Suggestion.—Prove that the radius of the midway circle equals $\frac{1}{2}(r + r')$, r and r' denoting the radii of the bases.

$AB : BE :: BC : BD$ - - - - - - (?)

$\therefore\ BD = EF$ - - - - - - - - (?)

Hence r exceeds r'' as much as r'' exceeds r' - - (?)

3. What is the convex surface of a right cone whose slant height is 40 feet; and the diameter of whose base is 8 feet?

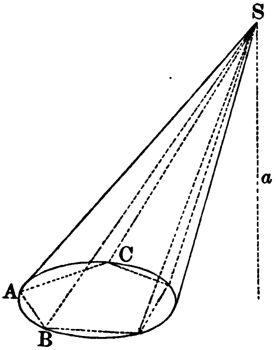
4. What is the entire surface of a right cone whose slant height is 18 feet, and the radius of whose base is 3 feet?

5. The slant height of a right cone is 20 feet; the circumference of its base, 12 feet. A cone is cut off whose slant height is 8 feet; required the convex surface of the frustum.

6. Find the entire surface of a frustum of a right cone, the slant height being 18 feet, and the radii of the bases 4 feet, and 3 feet.

THEOREM III.

The volume of any cone is equal to one-third the product of its base by its altitude.



Let S — A B C be a cone; then will its volume equal one-third the product of its base by its altitude.

For, inscribe within the cone a pyramid whose base is a regular polygon.

Then the volume of this pyramid equals one-third the product of its base by its altitude, whatever be the number of sides of its base.

But when the number of sides becomes infinite, the pyramid coincides in every respect with the cone.

Hence, the volume of the cone equals one-third the product of its base by its altitude. *Q. E. D.*

Cor.—*If* V *denotes the volume of the cone,* a *its altitude, and* r *the radius of its base, then*

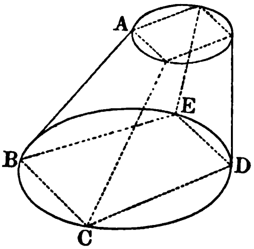
$$V = \tfrac{1}{3} a \pi r^2.$$

Scholium.—The student will observe that the volume of an oblique cone is found in the same manner as that of a right cone; but it is otherwise with respect to the surface; for this cannot be reduced to the measure of the sector of a circle, as in the case of the right cone, since all the lines drawn from the vertex to the base are not equal.

The right cone, whose altitude is to the radius of its base in the ratio of $\sqrt{2}$ to 1, possesses the remarkable property of having the greatest volume within a given surface.

THEOREM IV.

The volume of a frustum of a cone is equal to one-third the product of its altitude by the sum of its bases and a mean proportional between them.



Let A — B C E be a frustum of a cone; then will its volume be equal to one-third the product of its altitude by the sum of its bases and a mean proportional between them.

For, inscribe within it a frustum of a pyramid, whose bases are regular polygons. Then the volume of this frustum equals one-third of the product of its altitude by the sum of its bases and a mean proportional between them. - - - - - - - - - *Ch. XIV, Th. VI.*

But when the number of sides of its bases becomes indefinitely great, or infinite, the inscribed frustum coincides in every respect with the frustum of the cone.

Hence, the volume of the frustum of the cone equals one-third of the product of its altitude by the sum of its bases and a mean proportional between them. *Q. E. D.*

Cor. I.—*If* F *denotes the volume of the frustum,* B *and* b *its bases, and* a *its altitude, we have, as the algebraic expression of the theorem,*

$$F = \tfrac{1}{3}\, a\, (B + b + \sqrt{B\, b}).$$

Cor. II.—*If* r *denotes the radius of the lower base, and* r' *the radius of the upper, then*

$$B = \pi\, r^2, \textit{ and } b = \pi\, r'^2.$$

$$\therefore\quad F = \tfrac{1}{3}\, \pi\, a\, (r^2 + r'^2 + r\, r').$$

EXERCISES.

1. Cones having equal bases are proportional to their altitudes.

2. Similar right cones being those which are generated by similar right triangles revolving about homologous sides as axes, prove that similar right

cones are proportional to the cubes of their altitudes or to the cubes of the radii of their bases.

Suggestion.— $\frac{V}{V'}=\frac{\frac{1}{3}\pi r^2 a}{\frac{1}{3}\pi r'^2 a'}$.

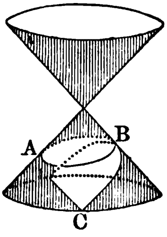
$\frac{a}{a'}=\frac{r}{r'}$.

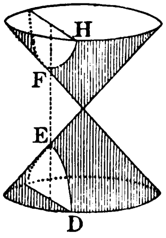
3. What is the volume of a cone whose altitude is 12½ feet, and the circumference of whose base is 10 feet?

4. A cask, consisting of two equal conic frustums united at their larger bases, has its bung diameter 28 inches, the head diameter 20 inches, and length 40 inches; required the capacity of the cask in gallons.

5. The volume of a cone is 8.83575 cubic feet, the altitude 15 feet; required the diameter of the base.

6. The convex surface of a frustum of a right cone is 378 square feet, the slant height 24 feet, and the area of the smaller base 28.2745; required the diameter of the greater base.





Note.—There are three curves, formed by the intersection of a plane with the surface of a cone, which are very celebrated by reason both of their application in the arts and sciences, and of the beauty of their forms and properties.

If a right cone with a circular base be cut by a plane, at an angle with the base *less* than that made by the generatrix, the section is an **Ellipse**, as A B.

If the cutting plane is *parallel* to the generatrix, the section is a **Parabola**, as B C.

If the cutting plane makes an angle with the base *greater* than that made by the generatrix, the section is a **Hyperbola**, as D E - F H.

These curves are known as **Conic Sections**; since they may be formed by passing a plane, in different positions, through a right circular *cone*.

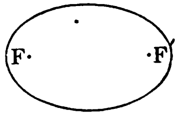
The distinguishing property of the Ellipse is, that the *sum* of the distances of any point in the bounding curve from two *fixed* points is constant.

The distinguishing property of the Parabola is, that the distance of any point in the curve from a fixed point is equal to its distance from a fixed right line.

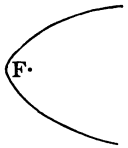
The distinguishing property of the Hyperbola is, that the *difference* of the distances of any point in either *branch* of the curve from two fixed points, is constant.

The fixed points are the **Foci** of the curves.

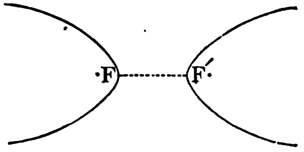
The heavenly bodies all move in conic sections.



ELLIPSE.



PARABOLA.



HYPERBOLA.

The path of each of the planets is an ellipse, of which the sun, as an attractive force, occupies one of the foci.

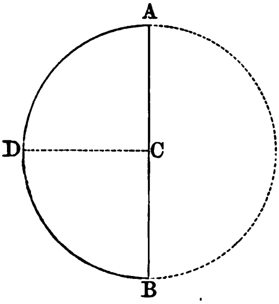
Most of the comets are found to move in curves which cannot be distinguished from parabolas; and several are supposed to move in hyperbolas.

These curves were first studied in the school of Plato, with no other motive however, than a simple love of intellectual activity, a matter of sublime speculation. He and his disciples had not thought of serving any practical end, and were reproached with the inutility of their inquiries. It was reserved for Galileo, and Kepler, and Newton, after two thousand years had rolled away, to give to these investigations a practical significance, by showing that these very curves with which Plato and the geometers of his school had *amused* themselves, were the paths of celestial movement: the first, to show that the path of a body projected obliquely in a vacuum is a *parabola;* the second, to discover that the planetary orbits are *ellipses;* the third, to demonstrate that a body which moves under the influence of a central attractive force, like that of the sun, whose intensity varies inversely as the square of the distance, must move in one of the conic sections, an Ellipse, a Parabola, or a Hyperbola.

CHAPTER XVII.

THE SPHERE.

DEFINITIONS.



If a semicircle A B D, whose centre is C, be supposed to revolve about its diameter A B as an axis, it will describe, as it revolves, a volume called a *sphere.* Hence,

1. A **Sphere** is a volume generated by the revolution of a semicircle about its diameter as an axis.

2. The path traced by the *semicircumference* is the boundary or **Surface** of the sphere.

Cor. I.—*The centre, radius, and diameter of the sphere are the same as those of the generating semicircle or circle.*

For the centre, radius, and diameter of the semicircle or circle are, it is evident, precisely the same for every position in the revolution.

Cor. II.—*Every point on the surface of a sphere is equally distant from the centre.*

For, in every position assumed by the revolving semicircle or circle, every point of the circumference is still equally distant from the centre C.

Cor. III.—*All radii and all diameters of the same sphere are equal.*

For these radii and diameters are the radii and diameters of the generating semicircle or circle.

Cor. IV.—*Spheres having equal radii are equal.*

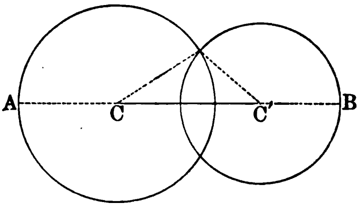
Cor. V.—*Two spheres are externally tangent if the distance between their centres is equal to the sum of their radii.*

For this would be true of their generating circles in every possible position of the revolution.

Cor. VI.—*Two spheres are tangent internally, if the distance between their centres is equal to the difference of their radii.*

Cor. VII.—*One sphere is wholly within another, if the distance between their centres is less than the difference of their radii.*

Cor. VIII.—*Two spheres are concentric, if the distance between their centres is zero.*



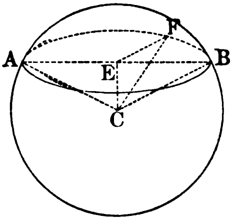
Cor. IX.—*Two spheres intersect, if the distance between their centres is less than the sum of their radii and greater than the difference.*

For this is true of the generating circles, C and C′, in every position of their revolution about A B.

Cor. X.—*Two spheres are external to each other, if the distance between their centres is greater than the sum of their radii.*

THEOREM I.

Any plane section of a sphere is a circle.



Let A B F be any plane section of the sphere whose centre is C; then A B F is a circle.

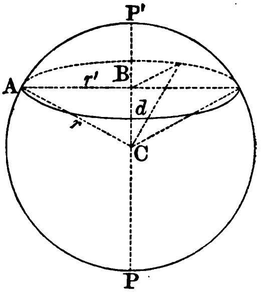
For, from the centre C, draw C E perpendicular to the plane of the section. Join C with different points in the boundary of the section, as A, B, and F.

Now, A, B, and F are points on the surface of the sphere; hence, C A, C B, and C F are radii, and therefore are equal.

Hence, E A, E F, and E B are equal; for, if these distances were *unequal,* the radii, C A, C B, and C F', would be unequal (*Ch. XII, Th. II*).

Hence, A, B and F are equally distant from E.

Hence, the section A B F is a circle (*Ch. IV, Th. VII; Ch. XII, Th. II, Sch.*). *Q. E. D.*



Cor. I.—*The centre of every plane section of a sphere is the foot of the perpendicular from the centre of the sphere to the section; and, hence, lies in a diameter of the sphere.*

As appears in the course of the preceding demonstration.

[This diameter of the sphere, perpendicular to the circle, is called the **Axis** of the circle; the extremities of the axis, as P and P', are called the **Poles** of the circle.]

Cor. II.—*All circles of a sphere equally distant from the centre of the sphere are equal.*

For, letting r denote the radius of the sphere, r' the radius of any section, and d the distance of the section from the centre of the sphere, we have from the right triangle A B C,

$$r' = \sqrt{r^2 - d^2}.$$

Similarly, if r'' and d' denote the radius and distance of any other section, we should have

$$r'' = \sqrt{r^2 - d'^2}.$$

But, by hypothesis, $d = d'$;

$$\therefore \quad d^2 = d'^2; \quad \therefore \quad \sqrt{r^2 - d^2} = \sqrt{r^2 - d'^2};$$

$$\therefore \quad r' = r''.$$

Hence, the radii being equal, the circles are equal.

Cor. III.—*A section is the largest possible, when it passes through the centre of the sphere.*

For, then $d = 0$, and

$$r' = \sqrt{r^2 - 0} = \sqrt{r^2} = r = \text{radius of the sphere.}$$

[A circle passing through the centre of the sphere, is called a **Great** circle; a circle not passing through the centre of the sphere is called a **Small** circle.]

Cor. IV.—*All great circles of a sphere are equal.*

For, for each such circle,

$$r' = \sqrt{r^2 - d^2} = \sqrt{r^2 - 0} = \sqrt{r^2} = r = \text{radius of sphere.}$$

Cor. V.—*The poles of a great circle are equally distant from every point in the circumference of the circle.*

For, the axis is perpendicular to the plane of the circle, and passes through the middle point of the circle.

Cor. VI.—*The poles of a small circle are unequally distant from the circumference* - - - - - - - - - - - (?)

Cor. VII.—*The arc of a great circle, drawn from the pole to the circumference of another great circle, is a quadrant.*

For, this arc is the measure of a central right angle.

Cor. VIII.—*A small circle is less as its distance from the centre of the sphere is greater.*

For, in the formula $r' = \sqrt{r^2 - d^2}$, as d increases, $r^2 - d^2$ decreases, and hence r' decreases.

Cor. IX.—*Any great circle of the sphere bisects it.*

For, the two parts into which the sphere is divided by a plane section through its centre, may be applied,[1] one to the other, so as to coincide; otherwise there would be points of the surface unequally distant from the centre.

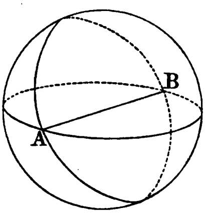
[The two equal parts into which a great circle divides the sphere, are called **Hemispheres.**]

Cor. X.—*Through any two points on the surface of a sphere one great circle can be made to pass, and only one, except when the two points are at the extremities of the same diameter.*

---

[1] Observe that the sphere here considered, is not a *physical* solid, but a *geometrical* one,—simply a portion of space.

Cor. XI.—*Two great circles of a sphere bisect each other.*



For, the planes of all great circles of a sphere must pass through the centre of the sphere.

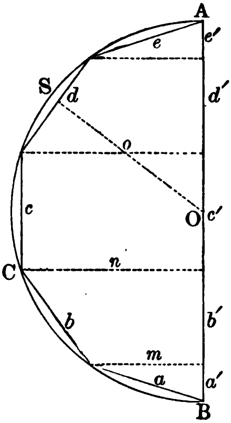
Hence the line of intersection of two great circles passes through the centre of the sphere, and is, therefore, both a diameter of the sphere and a common diameter of the two circles.

But a diameter of a circle bisects it; hence the common diameter bisects the two circles, or the circles bisect each other.

Cor. XII.—*The intersections of two circumferences of great circles on the surface of a sphere are 180° apart.*

For, these intersections, as A and B, are at the opposite extremities of a diameter.

DEFINITIONS.



1. If a semicircumference be divided into equal arcs, the chords of these arcs evidently form half of the perimeter of a regular inscribed polygon.

The half-perimeter is a **Regular Semi-perimeter**; the half-polygon, A B C, is a **Regular Semi-polygon**; the diameter A B is the **Axis** of the semi-polygon conceived as revolving about it; a', b', c', d', and e' are the *projections* on the axis, of the sides of the semi-perimeter; hence, the axis itself is the projection of the entire semi-perimeter.

2. If the semi-polygon revolves about its axis, the side a generates the convex surface of a right cone whose vertex is B, and whose base is the circle described by m; b, as also

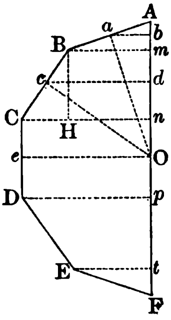
appears from the diagram, generates the convex surface of a frustum of a right cone whose bases are the circles described by m and n; c generates the convex surface of a right cylinder whose bases are the circles described by n and o; and O S, being the apothem, will trace a circle equal to the inscribed circle, if revolved in the plane of the semi-polygon, about the centre O.

3. **A Cylinder is Circumscribed about a Sphere** when its bases and convex surface are tangent to the sphere.

THEOREM II.

The surface generated by the revolution of a regular semi-polygon about its axis, is equal to the product of the axis by the circumference of the inscribed circle.

Let A F C be a regular semi-polygon, and A F its axis; then will the surface generated by the semi-perimeter revolving about A F, be equal to the product of A F by the circumference of the inscribed circle.



For, draw the apothems O a, O c, and O e, each of which equals r, the radius of the inscribed circle. Draw $a\,b$ and $c\,d$, B m, C n, and D p, perpendicular to A F. Draw B H parallel to A F.

Denote the surfaces generated by the respective sides of the semi-perimeter, by surface A B, surface B C, surface C D, *etc.*

Then,

(1) Surface $AB = 2\pi \cdot ab \cdot AB$ (*Ch. XVI, Th. II, Ex. 1*).

And (2) Surface $BC = 2\pi \cdot cd \cdot BC$ (*Ch. XVI, Th. II, Ex. 2*).

And (3) Surface $CD = 2\pi \cdot eO \cdot CD$ (*Ch. XV, Th. I*).

Now, the triangles AOa, Oab, and ABm are similar (*Ch. VIII, Th. V*). Hence, from ABm and Oab,

$Oa : AB :: ab : Am$;

or, $Oa : ab :: AB : Am$.

But, $Oa : ab :: 2\pi . Oa : 2\pi . ab$ - *Ch. X, Th. VII.*

$\therefore$ $2\pi . Oa : 2\pi . ab :: AB : Am$.

$\therefore$ (4) $2\pi . Oa . Am = 2\pi . ab . AB$.

Again, the triangles Ocd and BCH are similar (*Ch. VIII, Th. V*). Hence,

$Oc : BC :: cd : BH$;

or, $Oc : cd :: BC : BH$.

But, $Oc : cd :: 2\pi . Oc : 2\pi . cd$ - *Ch. X, Th. VII.*

And, $BH = mn$ - - - - - - - - - - - - (?)

$\therefore$ $2\pi . Oc : 2\pi . cd :: BC : mn$.

$\therefore$ (5) $2\pi . Oc . mn = 2\pi . cd . BC$.

Similarly, $2\pi . Oe . np = 2\pi . eO . CD$.

Substituting (4), (5) and (6), in (1), (2) and (3), and r for Oa, Oc, and Oe, we have

Surface $AB = 2\pi r . Am$;

Surface $BC = 2\pi r . mn$;

Surface $CD = 2\pi r . np$.

Similarly, Surface $DE = 2\pi r . pt$;

and Surface $EF = 2\pi r . tF$.

$\therefore$ Surface $ABCDEF = 2\pi r (Am + mn + np + pt + tF)$

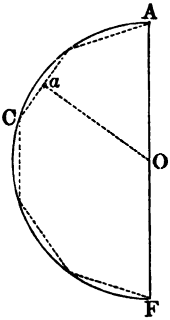
$= AF . 2\pi r$.

$=$ product of the axis by circumference of the inscribed circle. *Q. E. D.*

Scholium.—It is seen that the surface generated by any portion of the semi-perimeter is equal to its projection on the axis multiplied by the circumference of the inscribed circle.

THEOREM III.

The surface of a sphere is equal to the product of its diameter by the circumference of a great circle.



Let A C F be a semi-circumference, and A F its diameter; then will the surface generated by its revolution about A F, be equal to the product of A F by the circumference of a great circle.

For, inscribe within the semicircle a regular semi-polygon, whose apothem is O a.

Then the surface generated by the revolution of the semi-perimeter equals A F . 2 π . O a, whatever be the number of its sides.

But when the number of sides is infinite, the semi-perimeter coincides with the semi-circumference, and the apothem O a becomes equal to the radius O A.

Hence, the surface generated by the semi-circumference equals A F . 2 π . O A.

But O A is the radius of the generated sphere, and, hence, is the radius of a great circle.

Therefore, denoting the surface of the sphere by S,

$$S = AF \cdot 2\pi \cdot OA, = AF \cdot 2\pi r.$$

Q. E. D.

Cor. I.—*The surface of a sphere is equal to that of four great circles.*

For, $S = AF \cdot 2\pi \cdot OA = 2r \cdot 2\pi \cdot r = 4 \cdot \pi r^2$

= area of four great circles.

Cor. II.—*The surfaces of spheres are proportional to the squares of their radii.*

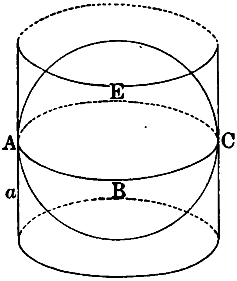
For, $S = 4\pi r^2$, and

$S' = 4\pi r'^2$; whence

$S : S' :: r^2 : r'^2$.

THEOREM IV.

The surface of a sphere is equal to the convex surface of the circumscribing cylinder.



For, the convex surface of the cylinder equals the circumference of its base multiplied by the altitude.

But the circumference of the base equals the circumference of A B C E, a great circle of the sphere; and the altitude of the cylinder equals the diameter of the sphere.

Hence, the convex surface of the cylinder equals circ. A B C E $\times D = 2\pi r \times 2r = 4\pi r^2$.

But the surface of the sphere equals $4\pi r^2$.

Hence, the surface of the sphere equals the convex surface of the circumscribing cylinder. *Q. E. D.*

Cor. I.—*The volume of the sphere is two-thirds of the circumscribing cylinder.*

For, the base of the cylinder being equal to a great circle of the sphere, and its altitude equal to the diameter of the sphere,

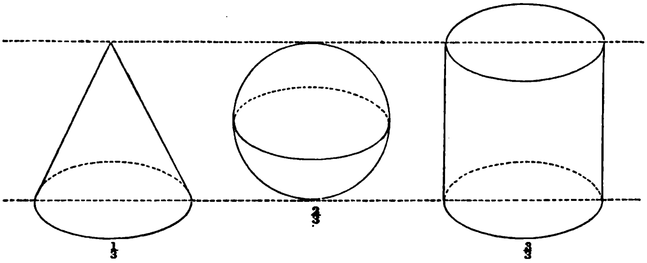
Volume of cylinder $= \pi r^2 . 2r = 2\pi r^3$ - *Ch. XV, Th. II.*

But, volume of sphere $= 4\pi r^2 . \frac{1}{3} r = \frac{4}{3}\pi r^3 = \frac{2}{3} . 2\pi r^3$

$=$ *two-thirds* of circumscribing cylinder.

Cor. II.—*If a cone and a cylinder have for their respective bases a great circle of a sphere, and for their altitude the diameter of that sphere, their volumes are to each other as*

1 : 2 : 3. - - - - - - - - - (?)



Cor. III.—*The volume of a hemisphere is equal to that of a cone having for its base a great circle of the sphere, and for its altitude the diameter of the sphere.* - - - - - - - - - - (?)

THEOREM V.

The volume of a sphere is equal to the product of its surface by one-third of its radius.

For the surface of the sphere may be conceived as made up of an infinite number of plane polygons; and if the vertices of these polygons be joined with the centre of the sphere, the sphere itself will be resolved into a corresponding number of pyramids, with the centre of the sphere as their common vertex, and the radius of the sphere as their common altitude.

Now the volume of *any* pyramid equals the product of its base by one-third of its altitude. - - - - *Ch. XIV, Th. V.*

Hence, the total volume of *all* the pyramids equals the sum of their bases multiplied by one-third of their common altitude.

But this total volume is the volume of the sphere, the sum of these bases is the surface of the sphere, and one-third of the common altitude is one-third of the radius of the sphere.

Hence, the volume of the sphere equals the product of its surface by one-third of its radius. *Q. E. D.*

Cor. I.—*The volume of a sphere is equal to* $\frac{4}{3}\pi r^3$.

For, denoting the volume by V, and the surface by S,

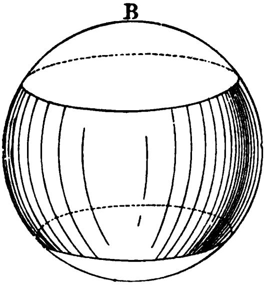
$$V = S \cdot \tfrac{1}{3} r.$$

But $S = 4\pi r^2$. - - - - - - *Th. III, Cor. I.*

$$\therefore \quad V = 4\pi r^2 \cdot \tfrac{1}{3} r = \tfrac{4}{3}\pi r^3.$$

Cor. II.—*The volume of a spherical sector is equal to the zone which forms its base multiplied by one-third of the radius.*

DEFINITIONS.



1. A **Zone** is a portion of the *surface* of a sphere included between the *circumferences* of two parallel circles.

2. The **Bases** of a zone are the bounding circumferences.

3. The **Altitude** of a zone is the perpendicular distance from the plane of one base to the plane of the other.

[If the plane of one base becomes tangent to the sphere, as at B, that base becomes a point, and the zone is said to have *one base.*]

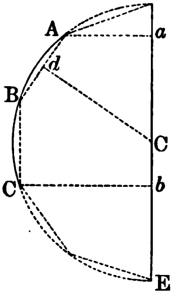
4. A **Spherical Segment** is a portion of a sphere bounded by a zone and the two parallel circles whose circumferences are the bases of the zone.

THEOREM VI.

The surface of a zone is equal to the product of its altitude by the circumference of a great circle.

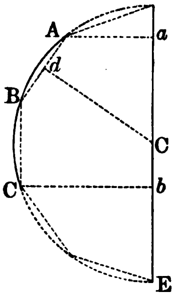
Let A B C be an arc, which, as the semicircle revolves about the diameter E F, generates a zone whose altitude is $a\,b$; then

$$\text{Surface } A\,B\,C = a\,b \,.\, 2\,\pi\,r.$$



For, inscribe within the semicircle a regular semi-polygon, and conceive the number of its sides continually to increase. When the number of sides becomes indefinitely great, they coincide, sensibly, with their subtended arcs, and hence may be taken to represent these arcs.

But the surface generated by any side is equal to the product of its projection on the axis by the circumference of the inscribed circle. (*Th. II, Sch.*)



Hence the surface generated by the subtended arc is equal to its projection on the axis multiplied by the circumference of the inscribed circle. This being true of *any* arc, is true of A B C.

Hence, surface A B C = its projection, $a\,b$, multiplied by $2\,\pi\,.\,\mathrm{C}\,d$.

But when the semi-polygon coincides with the semicircle,

$$\mathrm{C}\,d = \mathrm{C\,E} = r = \text{radius of the sphere.}$$

Hence, surface A B C $= a\,b\,.\,2\,\pi\,r$. *Q. E. D.*

EXERCISES.

1. Zones on the same sphere or on equal spheres, are proportional to their altitudes.

2. A zone is to the surface of the sphere as the altitude of the zone is to the diameter of the sphere.

3. Required the surface of a zone whose altitude is 2 feet, the diameter of the sphere being 12½ feet. *Ans. 78.54.*

4. Required the surface of a zone whose altitude is 24 feet, the radius of the sphere being 36 feet. *Ans. 5428.685.*

5. The axis of the earth is 7912 miles; the part of the axis intercepted by the North Frigid Zone is 327.19; by the North Temperate, 2053.468; by the North Torrid, 1575.338. What is the surface of each zone?

6. In the preceding example, what are the relative areas of the three zones?

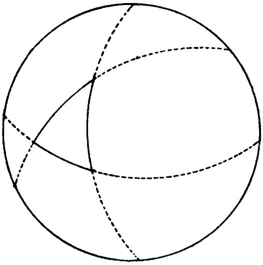
DEFINITIONS.

1. A **Spherical Angle** is an angle formed on the surface of a sphere by two intersecting arcs of great circles. Its measure is the same as that of the diedral angle included between the planes of its sides, and therefore it may be either *right* or *oblique*.

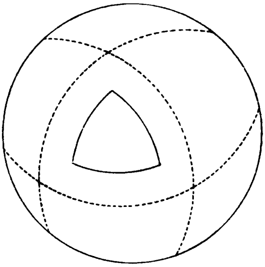
2. A **Spherical Polygon** is a portion of the surface of a sphere bounded by arcs of great circles. The bounding arcs are the *sides*, their consecutive inclinations are the *angles*, and their intersections are the *vertices*.

3. A **Diagonal** of a spherical polygon is an arc of a great circle joining the vertices of any two angles not consecutive.

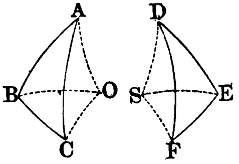
4. A **Spherical Triangle** is a spherical polygon of three sides. Spherical triangles, like plane triangles, are *right* or *oblique, scalene, isosceles,* or *equilateral.*



5. The **Polar** of a spherical triangle is the triangle formed by the intersection of three arcs of great circles described about the vertices of the given triangle as poles.



6. **Symmetrical Spherical Triangles** are such as have their parts equal, but arranged in different order, so that the triangles are incapable, as a rule, of superposition.



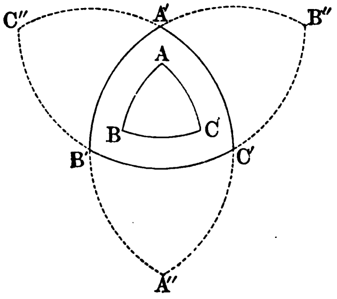
THEOREM VII.

If one triangle is the polar of another, the second is the polar of the first.

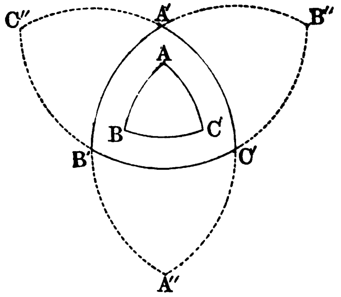
Let A′ B′ C′ be the polar of A B C: then will A B C be the polar of A′ B′ C′.

For, since B is the pole of A′ C′, and C of A′ B′, A′ is at a quadrant's distance from B and C. Hence A′ is the pole of B C.

In like manner it may be shown



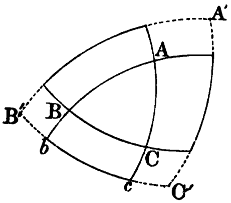
that B′ is the pole of A C, and C′ of A B. Hence A B C is the polar of A′ B′ C′. *Q. E. D.*



Scholium.—The arcs of great circles described about A, B, C, as poles, will, if sufficiently produced, form three triangles exterior to the polar. The polars, however, are distinguished by having their homologous vertices similarly situated with respect to the opposite sides.

THEOREM VIII.

In two polar triangles each angle of either is the supplement of the side lying opposite to it in the other.



Let A B C and A′ B′ C′ be two polar triangles: then will A, B, C, be supplements respectively of B′ C′, A′ C′, A′ B′; and A′, B′, C′, of B C, A C, A B.

For, let the sides of the interior triangle, produced if necessary, meet the sides of the exterior.

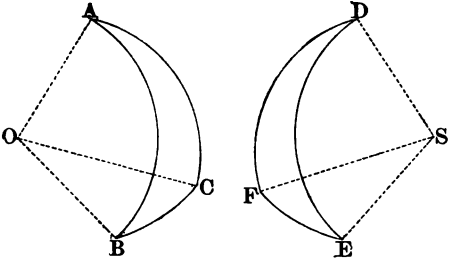
Angle $A = bc$, since A is the pole of B′ C′.
$B'c = 90°$, since B′ is the pole of A c.
$C'b = 90°$, since C′ is the pole of A b.
∴ $B'c + C'b = 180°$.
But $B'c + C'b = B'C' + bc$.
∴ $B'C' + bc = 180°$.
∴ $bc = 180° - B'C'$.
But $A = bc$.
∴ $A = 180° - B'C'$.
Similarly, $B = 180° - A'C'$,
$C = 180° - A'B'$,

$$A' = 180° - B\,C,$$
$$B' = 180° - A\,C,$$
$$C' = 180° - A\,B.$$ *Q. E. D.*

Cor.—*In either of two polar triangles, each side is the supplement of the angle opposite in the other.*

THEOREM IX.

If two triangles on equal spheres are mutually equilateral, they are mutually equiangular.

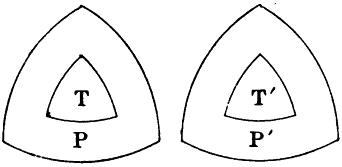


Let A B C and D E F be two triangles on equal spheres, having A B = D E, A C = D F, B C = E F: then will A = D, C = F, and B = E.

For, conceive the vertices to be joined with the centres of the spheres. The facial angles of the resulting triedrals are equal, each to each, because their measures are equal. Hence the planes of these equal angles are equally inclined to each other. That is, the angles of A B C are respectively equal to those of D E F. *Q. E. D.*

THEOREM X.

If two triangles on equal spheres are mutually equiangular, they are mutually equilateral.

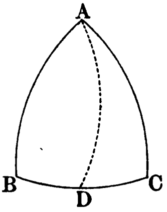


Let T and T′ be two spherical triangles which are mutually equiangular; then will they be mutually equilateral.

For, construct the polars P, P′; P and P′ being supplemental, are mutually equilateral, and therefore mutually equiangular. Hence T and T′, being supplemental, are mutually equilateral. *Q. E. D.*

THEOREM XI.

The angles opposite the equal sides of an isosceles spherical triangle are equal.



Let A B C be an isosceles spherical triangle, with A B = A C; then B = C.

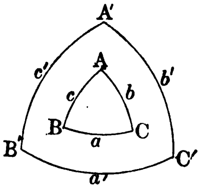
For, from the vertex A to the middle of the base, draw the arc of a great circle, A D. In the two triangles A B D, A D C,

A B = A C, by hypothesis;
A D = A D;
B D = D C, by construction;
∴ B = C. *Q. E. D.*

Cor.—*The arc drawn from the vertex of an isosceles spherical triangle to the middle of the base, bisects the vertical angle, is perpendicular to the base, and divides the triangle into two symmetrical triangles.*

THEOREM XII.

If two angles of a spherical triangle are equal, the triangle is isosceles.



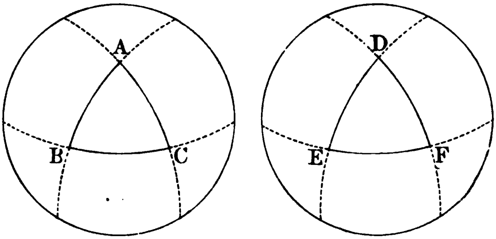
In the spherical triangle A B C, let B = C; then A B = A C.

For, draw the polar A′ B′ C′.

Now, $b' = 180° - B$,
$c' = 180° - C$.
but $B = C$;
∴ $b' = c'$;
∴ $B' = C'$.
Again, $b = 180° - B'$,
$c = 180° - C'$;
but $B' = C'$;
∴ $b = c$;
or, A B = A C. *Q. E. D.*

THEOREM XIII.

Symmetrical isosceles spherical triangles can be superimposed.

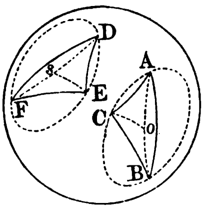


Let A B C and D E F be two symmetrical isosceles spherical triangles, with A B = A C, and D E = D F; and consequently A B = D E, A C = D F, angle A = angle D; then can they be superimposed.

For, applying A B to its equal D E, the extremity A at D, and B at E, with the angle C on the same side of D E as F, the arcs will coincide. Then, as angle A = angle D, A C will take the direction D F; and, since these arcs are equal, C will fall at F. Hence, B C will coincide with E F, as only one arc of a great circle can pass between E and F. *Q. E. D.*

THEOREM XIV.

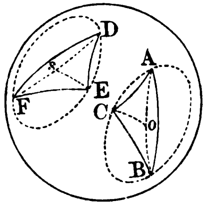
Symmetrical spherical triangles are equal.



Let A B C and D E F be two symmetrical spherical triangles, with A B = D F, A C = D E, B C = E F; then will they be equal.

For, through A, B, and C, D, E, and F, let small circles be passed, of which *o* and *s* are poles. Draw arcs of great circles from the poles to the vertices.

The small circles are equal because circumscribed about plane triangles whose sides, being chords of equal arcs, are respectively equal. Hence the six spherical distances are equal; that is,

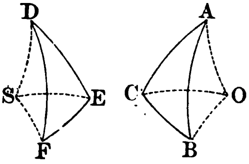


$o\mathrm{A} = s\mathrm{D} = o\mathrm{B} = s\mathrm{F} = o\mathrm{C} = s\mathrm{E}$.

But $\mathrm{AB} = \mathrm{DE}$, $\mathrm{CB} = \mathrm{EF}$, and $\mathrm{AC} = \mathrm{DE}$ by hypothesis.

$\therefore$ triangle $\mathrm{A}o\mathrm{B} = \mathrm{D}s\mathrm{F}$, $\mathrm{B}o\mathrm{C} = \mathrm{E}s\mathrm{F}$, $\mathrm{A}o\mathrm{C} = \mathrm{D}s\mathrm{E}$.

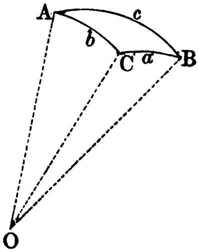
$\therefore$ $\mathrm{ABC} = \mathrm{DEF}$, since the two are composed of equal parts. *Q. E. D.*



Scholium.—If the poles of the small circles fall without the given triangles, A B C and D E F will be respectively equal to the sum of two of the partial triangles minus the third.

THEOREM XV.

Any side of a spherical triangle is less than the sum of the other two, and greater than their difference.



Let A B C be any spherical triangle on the sphere whose centre is O: then will any side be less than the sum, and greater than the difference, of the other two.

For, draw the radii O A, O B, O C, forming the triedral whose vertex is O. Since any facial angle of the triedral is less than the sum of the other two, the side of the triangle which measures it is less than the sum of the sides which measure the other facial angles. Therefore

$a < b + c$; whence $b > a - c$, $c > a - b$.

$b < a + c$; whence $a > b - c$, $c > b - a$.

$c < a + b$; whence $a > c - b$,. $b > c - a$. *Q. E. D.*

Cor.—*Any side of a spherical polygon is less than the sum of the other sides.*

THEOREM XVI.

The sum of the sides of a spherical triangle can have any value between 0 and a circumference.

For, these sides measure the facial angles of the corresponding triedral at the centre of the sphere. Hence their sum may be anything between 0 and the measure of four right angles. *Q. E. D.*

Cor.—*All the sides of a spherical triangle may be quadrants.*

Scholium I.—A spherical triangle is *quadrantal*, *bi-quadrantal* or *tri-quadrantal*, according as one of its sides is a quadrant, two are quadrants, or three are quadrants.

Scholium II—If we denote the sides of a spherical triangle by a, b, c, $a + b + c < 360°$.

THEOREM XVII.

The sum of the angles of a spherical triangle can have any value between two and six right angles.

For this sum is the same as that of the diedrals of a triedral having its vertex at the centre of the sphere. Now the limits of the latter sum are two and six right angles. Hence the sum of the angles of a spherical triangle can have any value between two right angles and six. *Q. E. D.*

Cor I.—*All the angles of a spherical triangle may be right.*

Cor. II.—*All the angles of a spherical triangle may be obtuse.*

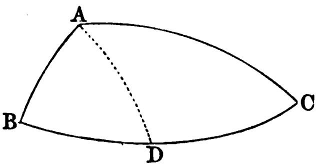
Scholium I.—A spherical triangle is *rectangular*, *bi-rectangular*, or *tri-rectangular*, according as it has one right angle, two right angles, or three right angles.

Scholium II.—A tri-rectangular triangle is one-eighth of the surface of the sphere. For the three great circles, each of which is perpendicular to the other two, divide the surface of the sphere into eight tri-rectangular triangles, mutually equilateral, and therefore equal.

Scholium III.—The amount by which the sum of the angles of a spherical triangle exceeds the sum of the angles of a plane triangle is called the *spherical excess.*

THEOREM XXII.

In any spherical triangle, the greater side is opposite the greater triangle.



In the spherical triangle ABC, let $A > B$; then $BC > AC$.

For, draw the arc of a great circle, AD, making $BAD = B$. Then,

$$AD = BD,$$

and $\quad AD + DC = BD + DC = BC.$

But, $\quad AD + DC > AC.$

$\therefore \quad BD + DC > AC;$

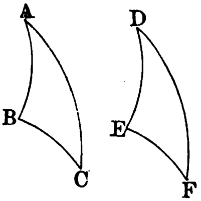
or, $\quad BC > AC.$ *Q. E. D.*

EXERCISES.

1. The angles of a triangle are 70°, 80°, and 100°; find the sides of its polar.

2. The sides of a triangle are 40, 90, and 125; find the angles of its polar.

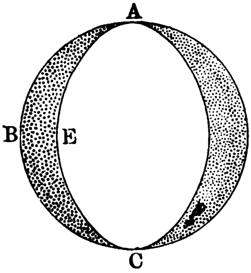
3. Find the lengths of the sides of the triangle in the preceding example, if the radius of the sphere is 6 feet.



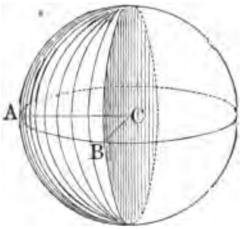
4. In any spherical triangle, the greater angle is opposite the greater side.

5. Spherical triangles are equal, (1) if they have two sides and the included angle of the one respectively equal to two sides and the included angle of the other; (2) if they have a side and two adjacent angles of one respectively equal to a side and two adjacent angles of the other.

DEFINITIONS.



1. A **Lune** is a portion of the surface of a sphere included between two semi-circumferences of great circles; as A B C E.



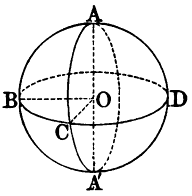
2. The **Angle of a Lune** is the angle included by the bounding semi-circumferences; or, which is the same, the measure of the diedral angle included between the planes of the great circles; as A C B = $\overset{\frown}{AB}$.

3. A **Spherical Wedge** is a portion of a sphere bounded by a lune and two great semicircles.

4. A **Spherical Sector** is a portion of sphere generated by a circular sector of the semicircle which generates the sphere.

THEOREM XIX.

A lune is to the surface of the sphere as the angle of the lune is to four right angles.



Let A B A′ C be a lune (L), whose angle is A, or B O C; S, the surface of the sphere; and B C D a great circle whose poles are A, A′; then will

L : S :: B C : B C D :: A : 4 R.

For, suppose

B C : B C D :: m : n.

If circle B C D be divided into n equal parts, m of these parts are contained in arc B C. If the poles A, A′, and the several points of division be joined, there will be formed 2 n equal isos-

Cor. III.—*The volume of a hemisphere is equal to that of a cone having for its base a great circle of the sphere, and for its altitude the diameter of the sphere.* - - - - - - - - - - (?)

THEOREM V.

The volume of a sphere is equal to the product of its surface by one-third of its radius.

For the surface of the sphere may be conceived as made up of an infinite number of plane polygons; and if the vertices of these polygons be joined with the centre of the sphere, the sphere itself will be resolved into a corresponding number of pyramids, with the centre of the sphere as their common vertex, and the radius of the sphere as their common altitude.

Now the volume of *any* pyramid equals the product of its base by one-third of its altitude. - - - - *Ch. XIV, Th. V.*

Hence, the total volume of *all* the pyramids equals the sum of their bases multiplied by one-third of their common altitude.

But this total volume is the volume of the sphere, the sum of these bases is the surface of the sphere, and one-third of the common altitude is one-third of the radius of the sphere.

Hence, the volume of the sphere equals the product of its surface by one-third of its radius. *Q. E. D.*

Cor. I.—*The volume of a sphere is equal to* $\frac{4}{3}\pi r^3$.

For, denoting the volume by V, and the surface by S,

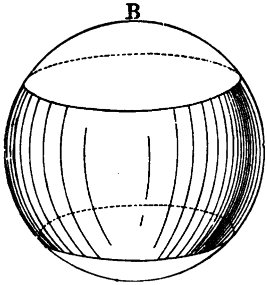
$$V = S \cdot \tfrac{1}{3} r.$$

But $S = 4\pi r^2$. - - - - - *Th. III, Cor. I.*

$$\therefore \quad V = 4\pi r^2 \cdot \tfrac{1}{3} r = \tfrac{4}{3}\pi r^3.$$

Cor. II.—*The volume of a spherical sector is equal to the zone which forms its base multiplied by one-third of the radius.*

DEFINITIONS.



1. A **Zone** is a portion of the *surface* of a sphere included between the *circumferences* of two parallel circles.

2. The **Bases** of a zone are the bounding circumferences.

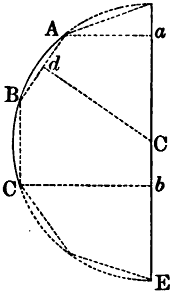
3. The **Altitude** of a zone is the perpendicular distance from the plane of one base to the plane of the other.

[If the plane of one base becomes tangent to the sphere, as at B, that base becomes a point, and the zone is said to have *one base.*]

4. A **Spherical Segment** is a portion of a sphere bounded by a zone and the two parallel circles whose circumferences are the bases of the zone.

THEOREM VI.

The surface of a zone is equal to the product of its altitude by the circumference of a great circle.



Let A B C be an arc, which, as the semicircle revolves about the diameter E F, generates a zone whose altitude is $a\,b$; then

$$\text{Surface } A\,B\,C = a\,b\,.\,2\,\pi\,r.$$

For, inscribe within the semicircle a regular semi-polygon, and conceive the number of its sides continually to increase. When the number of sides becomes indefinitely great, they coincide, sensibly, with their subtended arcs, and hence may be taken to represent these arcs.

But the surface generated by any side is equal to the product of its projection on the axis by the circumference of the inscribed circle. (*Th. II, Sch.*)

Cor. III.—*The volume of a hemisphere is equal to that of a cone having for its base a great circle of the sphere, and for its altitude the diameter of the sphere.* - - - - - - - - - - (?)

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Hence, the total volume of *all* the pyramids equals the sum of their bases multiplied by one-third of their common altitude.

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Hence, the volume of the sphere equals the product of its surface by one-third of its radius. *Q. E. D.*

Cor. I.—*The volume of a sphere is equal to* $\frac{4}{3}\pi r^3$.

For, denoting the volume by V, and the surface by S,

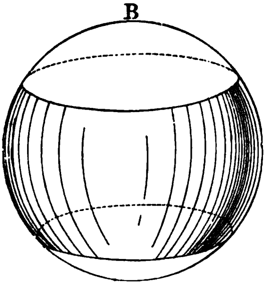
$$V = S \cdot \tfrac{1}{3} r.$$

But $S = 4\pi r^2$. - - - - - *Th. III, Cor. I.*

$$\therefore \quad V = 4\pi r^2 \cdot \tfrac{1}{3} r = \tfrac{4}{3}\pi r^3.$$

Cor. II.—*The volume of a spherical sector is equal to the zone which forms its base multiplied by one-third of the radius.*

DEFINITIONS.



1. A **Zone** is a portion of the *surface* of a sphere included between the *circumferences* of two parallel circles.

2. The **Bases** of a zone are the bounding circumferences.

3. The **Altitude** of a zone is the perpendicular distance from the plane of one base to the plane of the other.

[If the plane of one base becomes tangent to the sphere, as at B, that base becomes a point, and the zone is said to have *one base.*]

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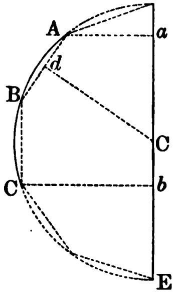
THEOREM VI.

The surface of a zone is equal to the product of its altitude by the circumference of a great circle.

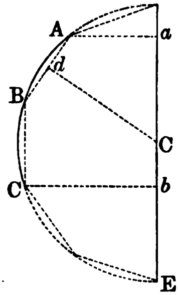


Let A B C be an arc, which, as the semicircle revolves about the diameter E F, generates a zone whose altitude is $a\,b$; then

$$\text{Surface } A\,B\,C = a\,b\,.\,2\,\pi\,r.$$

For, inscribe within the semicircle a regular semi-polygon, and conceive the number of its sides continually to increase. When the number of sides becomes indefinitely great, they coincide, sensibly, with their subtended arcs, and hence may be taken to represent these arcs.

But the surface generated by any side is equal to the product of its projection on the axis by the circumference of the inscribed circle. (*Th. II, Sch.*)



Hence the surface generated by the subtended arc is equal to its projection on the axis multiplied by the circumference of the inscribed circle. This being true of *any* arc, is true of A B C.

Hence, surface A B C = its projection, $a\,b$, multiplied by $2\,\pi\,.\,C\,d$.

But when the semi-polygon coincides with the semicircle,

$$C\,d = C\,E = r = \text{radius of the sphere.}$$

Hence, surface A B C = $a\,b\,.\,2\,\pi\,r$. *Q. E. D.*

EXERCISES.

1. Zones on the same sphere or on equal spheres, are proportional to their altitudes.

2. A zone is to the surface of the sphere as the altitude of the zone is to the diameter of the sphere.

3. Required the surface of a zone whose altitude is 2 feet, the diameter of the sphere being 12½ feet. *Ans. 78.54.*

4. Required the surface of a zone whose altitude is 24 feet, the radius of the sphere being 36 feet. *Ans. 5428.685.*

5. The axis of the earth is 7912 miles; the part of the axis intercepted by the North Frigid Zone is 327.19; by the North Temperate, 2053.468; by the North Torrid, 1575.338. What is the surface of each zone?

6. In the preceding example, what are the relative areas of the three zones?

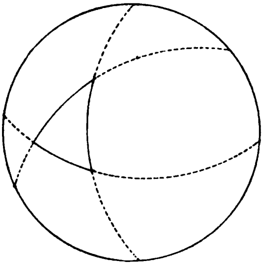
DEFINITIONS.

1. A **Spherical Angle** is an angle formed on the surface of a sphere by two intersecting arcs of great circles. Its measure is the same as that of the diedral angle included between the planes of its sides, and therefore it may be either *right* or *oblique*.

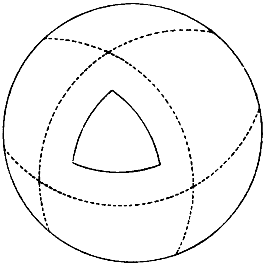
2. A **Spherical Polygon** is a portion of the surface of a sphere bounded by arcs of great circles. The bounding arcs are the *sides*, their consecutive inclinations are the *angles*, and their intersections are the *vertices*.

3. A **Diagonal** of a spherical polygon is an arc of a great circle joining the vertices of any two angles not consecutive.

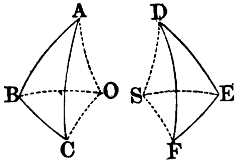
4. A **Spherical Triangle** is a spherical polygon of three sides. Spherical triangles, like plane triangles, are *right* or *oblique*, *scalene*, *isosceles*, or *equilateral*.



5. The **Polar** of a spherical triangle is the triangle formed by the intersection of three arcs of great circles described about the vertices of the given triangle as poles.

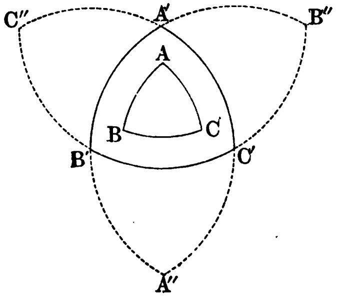


6. **Symmetrical Spherical Triangles** are such as have their parts equal, but arranged in different order, so that the triangles are incapable, as a rule, of superposition.



THEOREM VII.

If one triangle is the polar of another, the second is the polar of the first.

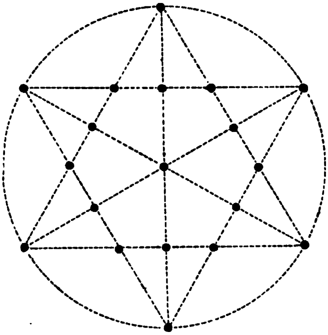


Let $A'B'C'$ be the polar of ABC: then will ABC be the polar of $A'B'C'$.

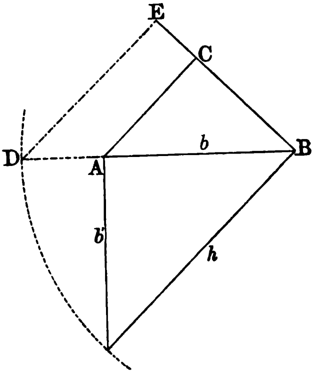
For, since B is the pole of $A'C'$, and C of $A'B'$, A' is at a quadrant's distance from B and C. Hence A' is the pole of BC.

In like manner it may be shown

22. The volume of a cone is c; the radius of its base is r: required the volume of the circumscribed sphere.



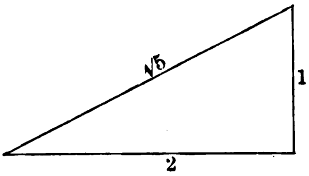
23. To plant nineteen trees so that they shall constitute nine straight rows, with five trees in each row.



24. Given any triangle, to construct a similar one of twice the area.

Suggestion.—$h^2 = b'^2 + b^2 = 2\,b^2$. D E is parallel to A C. Similar triangles are proportional to the squares of what?

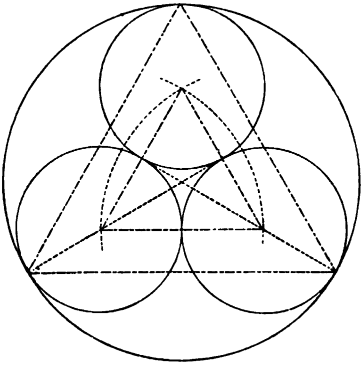
25. To construct a right triangle whose base shall be 2 and the hypotenuse 4.



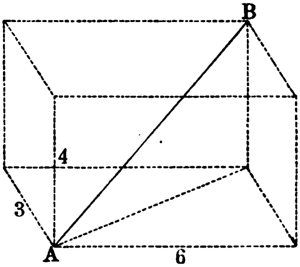
26. On a given straight line, to construct an equilateral triangle, a square, and a hexagon.

27. To construct a square whose area shall be five square inches.

28. With a given radius, to describe three equal circles which shall touch one another, and then to describe another circle which shall touch them all three.



29. The length, width, and height of a room are 6, 3, and 4; required its diagonal, A B.



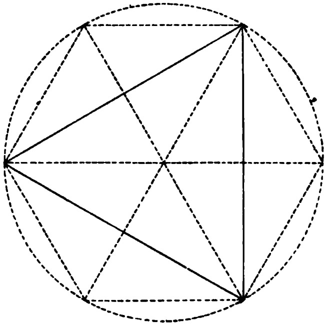
30. Find the length of an arc of 28° on a circumference whose radius is 10.

31. Find the length of an arc of 45° 45′, the radius being 15.

32. Find the area of a sector whose arc is 30° 30′, and whose radius is 24.

33 The entire surface of a cube is a; required its volume, and its diagonal.

34. The area of an equilateral triangle is equal to one-fourth of the square of one side, multiplied by the square root of 3.



35. Find the entire surface of a right prism whose altitude is 16, and whose base is an equilateral triangle, each side of which is 6.

36. Find the entire surface of a right prism whose altitude is 18, and whose base is a regular hexagon, each side of which is 5.

37. Find the entire surface of a right pyramid whose slant height is 16 feet, and whose base is three feet square.

[illegible], Limerick